

PERIODIC MEAN ANNUAL INCREMENT  
AND THE DERIVATIVE  
IN GROWTH PREDICTION

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### ABSTRACT

Estimation of growth rates by the  $\Delta$ -process and by differential calculus is briefly examined. Five techniques of summing increments are compared. It is demonstrated that both the algebraic and the calculus approaches can give equally good estimates of the growth and yield of forest stands. Calculus methods may, however, be preferred: they are simpler and likely to be more consistent and compatible.

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## INTRODUCTION

Since differential and integral calculus methods are becoming increasingly popular in growth and yield studies, a brief examination of the relationship between the periodic mean annual increment (PMAI) and the derivative of growth functions is perhaps appropriate.

The algebraic derivation (or  $\Delta$ -process) of the stand-volume increment has already been described in detail by Evert (1964). Forestry literature deals mostly with growth prediction in terms of the  $\Delta$ -process and, consequently, increment is estimated generally on an annual basis. In recent studies, however, the advantages of using differential and integral calculus methods have been recognized, and thus growth rate is expressed as an instantaneous rate of change with respect to age or time (Buckman 1962, Clutter 1963, Curtis 1967).

## GROWTH RATE

The rate of growth of forest stands may be expressed in several ways. For example, mean annual increment (MAI) refers to the average annual growth during the period between stand establishment and a particular age. PMAI, on the other hand, expresses the average annual growth during any definite period of time, such as 10 years. In terms of differential calculus, the derivative of a growth function is comparable to both the MAI and the PMAI.

The relationship between the PMAI and the derivative may be illustrated briefly through the stand-volume formula:

$$V = F B H \quad \text{Eq. 1}$$

where:  $V$  is the total volume per unit area

$F$  is the mean form factor of the stand

$B$  is the total basal area at breast height, per unit area

$H$  is the mean height of the stand.

So, if  $\Delta$  denotes increment (in this case PMAI) in each of the variables, then the PMAI in volume is:

$$\Delta V = B H \Delta F + F H \Delta B + F B \Delta H + F \Delta B \Delta H + B \Delta F \Delta H + H \Delta F \Delta B + \Delta F \Delta B \Delta H \quad \text{Eq. 2}$$

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If Equation 1 is differentiated with respect to age, the derivative is:

$$\frac{dV}{dA} = B_H \frac{dF}{dA} + F_H \frac{dB}{dA} + F_B \frac{dH}{dA} \quad \text{Eq. 3}$$

The difference between Equations 2 and 3 lies in the concept of the limit

$$\frac{dV}{dA} = \lim_{\Delta A \rightarrow 0} \frac{\Delta V}{\Delta A}$$

i.e.,  $\Delta A \rightarrow 0$  (is approaching zero) for the derivative, whereas for PMAI,  $\Delta A = 1$ . When  $\Delta A$  refers to a relatively short period of time, it is reasonable to assume that in established stands the change in form is negligible ( $\Delta F = 0$ ), and thus the last two equations are:

$$\Delta V = F_H \Delta B + F_B \Delta H + F \Delta B \Delta H \quad \text{Eq. 2'}$$

$$\frac{dV}{dA} = F_H \frac{dB}{dA} + F_B \frac{dH}{dA} \quad \text{Eq. 3'}$$

The terms  $\Delta B$ ,  $\Delta H$ , and  $\Delta F$  are obtained by measuring or estimating the average annual growth in the particular variable between two ages or two points in time. In order to arrive at values for  $dF/dA$ ,  $dB/dA$ , and  $dH/dA$ , it is necessary to express their respective variables as continuous functions of age or time, i.e.,

$$F = f_1(A) \quad \text{and} \quad F = f_1(\text{Time})$$

$$B = f_2(A) \quad B = f_2(\text{Time})$$

$$H = f_3(A) \quad H = f_3(\text{Time})$$

## YIELD

Cumulative yield is the sum of annual increments since stand establishment. Therefore, volume at a number of years ( $n$ ) after a specified age ( $a$ ) may be obtained by different ways, such as:

$$a) V_{a+n} = V_a + \sum_{i=a}^{a+(n-1)} \Delta V_i \quad \text{Eq. 4}$$

$$b) V_{a+n} = \int_0^{a+n} dV' dA \quad \text{Eq. 5}$$

$$c) V_{a+n} = \sum_{i=1}^{a+n} (MAI) \quad \text{Eq. 6}$$

$$d) V_{a+n} = V_a + \int_a^{a+n} dV \, dA \quad \text{Eq. 7}$$

where:  $dV'$  is the differential of a volume function ( $V = f(A)$  or  $V = f(\text{Time})$ ), which refers to the total period between stand establishment and  $a+n$

$dV$  is the differential of a volume function ( $V = f(A)$  or  $V = f(\text{Time})$ ), which refers only to the period between a particular stand age and  $a+n$ .

In general, Equations 5 and 6 are useful for management and economic planning, when growth simulation is performed on a hypothetical forest. Equations 4 and 7, on the other hand, illustrate a practical approach for predicting future yields from known stand volumes.  $V_a$  is thus obtained by any conventional forest inventory technique.

In the past, stand increment was often estimated for a definite period of time, such as 10 years, from permanent-sample-plot data. The use of PMAI's or derivatives of stand-component functions, however, facilitates yield prediction in a simple manner, especially when permanent sample plots have varying periods of remeasurement.

## TESTS

One hundred and thirty-six permanent sample plots with a 5- to 10-year remeasurement period were used to test five techniques of summing stand-volume increments. First, volume yield was predicted to the end of the remeasurement period; then observed and predicted volumes were tested for difference by treating the values as paired data. Increments in F, B, and H were estimated by PMAI functions of the form

$$\Delta B_1 = a + b_1 B_a + b_2 A_a^{-1}$$

$$\Delta H_i = a' + b'H_a$$

$$\Delta F_i = a'' + b_1''\Delta B_i + b_2''\Delta H_i$$

where:  $a$ ,  $b_1$ ,  $b_2$ ,  $a'$ ,  $b'$ ,  $a''$ ,  $b_1''$ , and  $b_2''$  are estimated regression coefficients, and  $A$  is stand age.

The tests are as follows.

Test 1: Iterative yearly build-up of the PMAI of volume in the form of Equation 4, where

$$\begin{aligned}\Delta V_i = & B_a H_a \Delta F_i + F_a H_a \Delta B_i + F_a B_a \Delta H_i + F_a \Delta B_i \Delta H_i + B_a \Delta F_i \Delta H_i \\ & + H_a \Delta F_i \Delta B_i + \Delta F_i \Delta B_i \Delta H_i\end{aligned}$$

N.B.  $V_{a+n}$  of the first step is equated with  $V_a$  of the second step and, similarly,  $F_a + \Delta F_i$ ,  $B_a + \Delta B_i$ ,  $H_a + \Delta H_i$  of the first step are equated in the second step with  $F_a$ ,  $B_a$ , and  $H_a$ , respectively.

Test 2: Iterative yearly build-up of the PMAI of volume in the form of Equation 4, where

$$\Delta V_i = F_a H_a \Delta B_i + F_a B_a \Delta H_i + F_a \Delta B_i \Delta H_i$$

i.e., the mean-stand-form factor is held constant.

Test 3: Iterative yearly build-up of the PMAI of volume in the form of Equation 4, where

$$\Delta V_i = F_a H_a \Delta B_i + F_a B_a \Delta H_i$$

i.e., the mean-stand-form factor is held constant, and the derivative is estimated by PMAI.

Test 4: Iterative yearly build-up of the PMAI of volume in the form of Equation 4, where

$$\Delta V_i = F_a H_a \Delta B_i + F_a B_a \Delta H_i + B_a H_a \Delta F_i$$

i.e., terms containing more than one  $A$  are omitted.

Test 5: Integration (by approximation) of the stand-volume function:

$$V_{a+n} = V_a + \int_a^{a+n} dV \, dA$$

and  $V_a$  is expressed in the form of a multiple regression

$$V_a = a''' + b_1''' (B_a H_a) + b_2''' B_a + b_3''' (H_a A_a^{-1})$$

where  $a'''$ ,  $b_1'''$ ,  $b_2'''$ , and  $b_3'''$  are estimated regression coefficients.

Therefore, assuming constant form for the stand over the period, the total periodic increment is estimated as

$$\begin{aligned} \int_a^{a+n} dV \, dA &= b_1''' (B_{a+n} H_{a+n} - B_a H_a) + b_2''' (B_{a+n} - B_a) \\ &+ b_3''' (H_{a+n} A_{a+n}^{-1} - H_a A_a^{-1}) \end{aligned}$$

$$\text{where: } B_{a+n} = B_a + n\Delta B_i$$

$$H_{a+n} = H_a + n\Delta H_i$$

The results of these tests are tabulated in Table 1.



Table 1. Results of five techniques of summing stand-volume increments.

| Test | $\bar{d}$<br>cu ft/acre | $s$<br>cu ft/acre | $s_{\bar{d}}$<br>cu ft/acre |
|------|-------------------------|-------------------|-----------------------------|
| 1    | 0.9                     | 153.7             | 13.2                        |
| 2    | 17.0                    | 154.0             | 13.2                        |
| 3    | 15.0                    | 154.0             | 13.2                        |
| 4    | -0.8                    | 153.8             | 13.2                        |
| 5    | 17.0                    | 147.7             | 12.7                        |

where:  $\bar{d}$  is the mean difference between predicted and observed values,

$s$  is the standard deviation of  $\bar{d}$ , and

$s_{\bar{d}}$  is the standard error of  $\bar{d}$ ;

when: the mean observed total increment is 405.2 cu ft/acre,

the weighted mean-growth period is 8.6 years,

the range in the stand age is 32 to 135 years, and

the species is jack pine (*Pinus banksiana* Lamb.).

## DISCUSSION

The foregoing five tests illustrate a number of useful points. With Test 1 as a basis for comparison, Test 2 indicates that the assumption of no increase in the mean-stand-form factor over a relatively short period of time (up to 10 years in this study) may be accepted. Test 3 shows that the derivative may be approximated by the PMAI ( $\Delta F_i = 0$  is also assumed). Test 4 confirms a suggestion made by Wiedemann (Evert 1964) that with the exception of young stands, increment values for short periods are relatively small, and terms containing two or more deltas may therefore be dropped from Equation 2. Test 5 utilizes a stand-volume regression surface, and at the same time integration is attained by approximation. Again the final result is not changed significantly. Yield estimation by Equation 7 is obtained by the integration of a relevant volume function (one age). When a reliable volume function on age is not available, some of the basic principles of integral calculus may still be applied for the sake of consistency, and thus integration can be obtained by approximation as shown in Test 5.

In general, the  $\Delta$ -process and the calculus approach both give comparable results. Perhaps the greatest advantages of using differential equations are that they facilitate the location of conditions under which growth is maximized (or minimized) in a simple way, and that they are more consistent in their use. Permanent sample plots are still needed until a reliable method of predicting mortality is discovered. However, under certain conditions, B and H may be expressed as a reliable function of age or time, through either stem analysis or repeated measurements of temporary sample plots over a relatively short period of time. The differentiation of such functions with respect to age or time will then make it possible to estimate the instantaneous rate of change in the respective dependent variable.

### CONCLUSIONS

It is concluded that both the  $\Delta$ -process (in terms of annual increment) and differential and integral calculus methods can give equally good estimates of the growth and yield of forest stands. Although the need for permanent sample plots is recognized, it is not necessary to set the remeasurement period to 10 or 15 years. Sample plots with varying periods of remeasurement can be incorporated into a growth and yield study in a simple manner through the PMAI or the derivative. Calculus methods are preferred to the  $\Delta$ -process because they are more flexible (with regard to the location of conditions under which growth is maximized or minimized) and are likely to be more consistent and compatible in their use.

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