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Formulation of the form-class taper curves

I.S. Alemdag

Information Report PI-X-22
Petawawa National Forestry Institute

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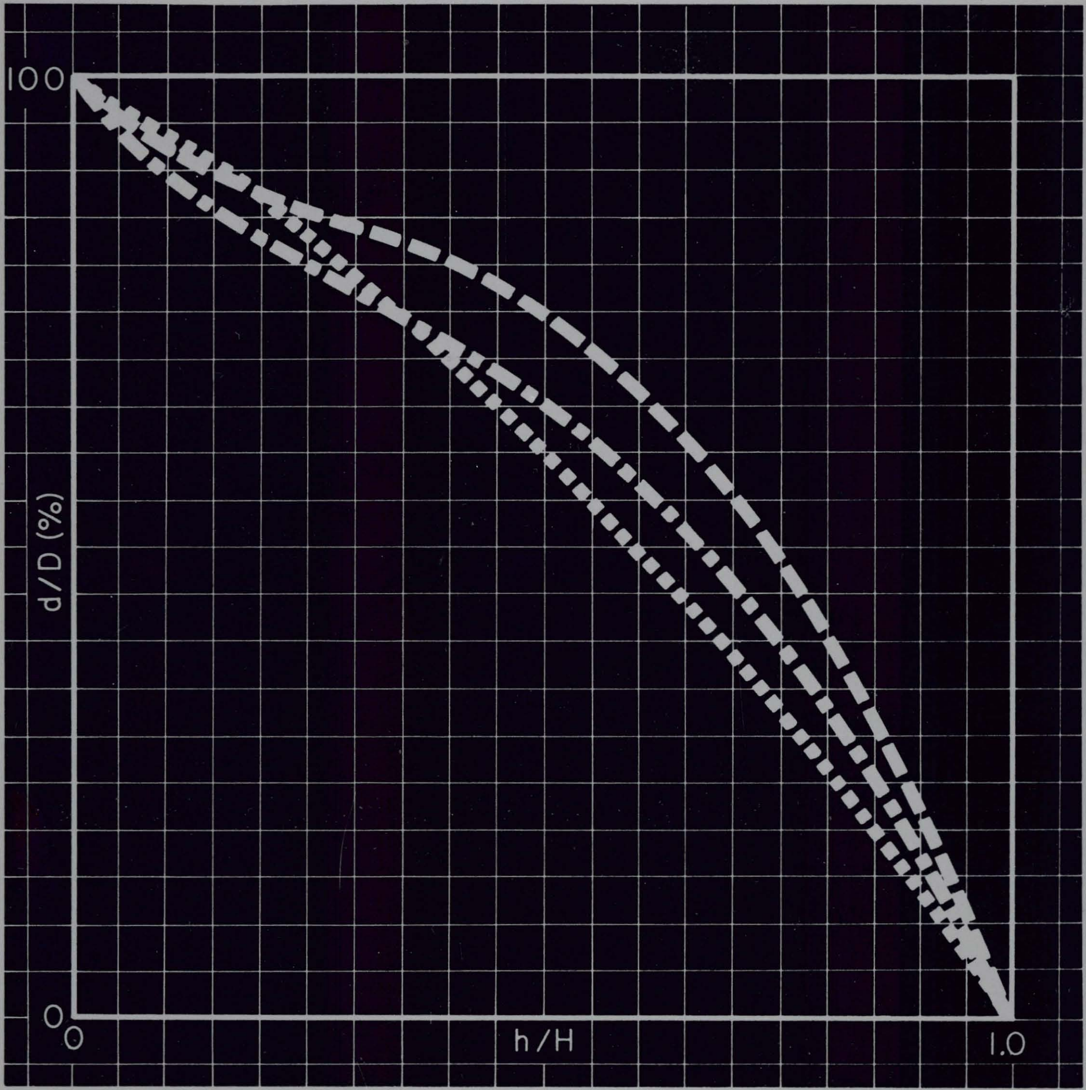

PETAWAWA NATIONAL FORESTRY INSTITUTE

The Petawawa National Forestry Institute was formed on April 1, 1979, as the result of an amalgamation of the former Petawawa Forest Experiment Station, which had existed for more than 60 years, with the Ottawa-based Forest Management and Forest Fire Research institutes.

In common with the rest of the Canadian Forestry Service, the Petawawa National Forestry Institute has as its objective the promotion of better management and wiser use of Canada's forest resource to the economic and social benefit of all Canadians. Because it is a national institute, particular emphasis is placed on research into problems that transcend regional boundaries or that require special expertise and expensive equipment that cannot be duplicated in CFS regional establishments. Such research is often performed in close cooperation with staff of the regional centres or provincial forest services.

Research at the institute is in two main areas:

FIRE RESEARCH AND REMOTE SENSING. Every year in Canada large areas of productive forest are destroyed by fire. Research concentrates on studies of forest fire behaviour, the development of new methods of fire control, the evaluation of fire-fighting equipment and retardants, and the development of computerized fire management systems that are rapidly finding applications with fire-fighting agencies across the country. The environmental and economic impact of forest fires and the use of fire as a silvicultural tool for intensive forest management are also studied.

In remote sensing, investigations are made into the application of modern satellite and airborne remote sensing systems to forestry problems. In this respect, the ARIES digital image analysis system is proving invaluable.

INTENSIVE FOREST MANAGEMENT. As Canada moves into more intensive management of its forests to meet expected increases in demand for this important resource, the role of this program will become increasingly important. An extensive reforestation program will require a steady supply of high-quality seed of the desired species. Improved growing stock, obtained through tree breeding and forest genetics research, is highly desirable. Increased emphasis is being placed on using the entire above-ground portion of the tree (biomass), but the effect on the environment of this and other forms of intensive management has to be carefully monitored. Biotechnological methods of improving yield while maintaining site productivity are being investigated.

In support of its research programs, the institute has at its disposal a 98-km^2 area of forest in the northern part of the Petawawa military reserve. Records of experiments and sample plots have been maintained since the 1920s. The forest also serves as a field laboratory for students from local schools, and a visitor centre is operated during the summer months.

FORMULATION OF THE FORM-CLASS TAPER CURVES

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Contents

Tables

Figures

FORMULATION OF THE FORM-CLASS TAPER CURVES

Abstract
The equations were developed for the free-hand form-class taper curves that were prepared by the Canadian Forestry Service many years ago. The purpose was to eliminate errors due to interpolations and to make them adaptable for computer use as well as for metric usage.

Résumé
Les équations ont été développées pour les courbes à main levee de défilement exprimées en quotient de forme. Ces courbes avaient été préparées par le service canadien des forêts plusieurs années auparavant. Le but était d'éliminer les erreurs dues aux interpolations et de les rendre adaptables aussi bien pour l'ordinateur que pour utilisation dans le système métrique.

INTRODUCTION

Form-class taper tables for various tree species were first published by the Canadian Forestry Service in 1930, together with form-class volume tables (Anon. 1930). These tables were revised in 1948 (Anon. 1948). For each tree species, or size or age group within a species, free-hand taper curves were given for two or more form classes. The species studied were balsam fir (Abies balsamea (L.) Mill.), jack pine (Pinus banksiana Lamb.), lodgepole pine (Pinus contorta Dougl. var. latifolia Engelm.), red pine (Pinus resinosa Ait.), eastern white pine (Pinus strobus L.), white spruce (Picea glauca (Moench) Voss), red spruce (Picea rubens Sarg.) and black spruce (Picea mariana (Mill.) B.S.P.). The spruces were studied as one group.

In these taper tables, taper ratios are provided only for some relative heights. When a ratio at a point on the stem other than at these relative heights is needed, an interpolation is required. That can be expected to introduce a source of error into the investigations. Bonnor (1968) demonstrated that a straight-line interpolation between the two measurement points on the stem results in significant errors. Therefore, the purpose of this report is to formulate these curves in order to eliminate this interpolation problem, and at the same time make them adaptable to the use of digital computers. While formulating these tables, metric conversion was considered, but was found to be unnecessary.[1] This will be discussed in detail later in this report.

BACKGROUND INFORMATION

Taper, by definition, is "...the decrease in thickness, generally in terms of diameter, of a tree stem or log from the base upwards" (Ford-Robertson 1971). The curves or tables incorporating these decreases are used primarily for constructing total-stem or merchantable volume tables, and to provide a good picture of the longitudinal profile of stem. Along this line Belyea (1931) states: "The definite endeavor of a taper table is to demonstrate the average form of trees as

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I.S. Alemdag is a research scientist at the Petawawa National Forestry Institute, Canadian Forestry Service, Environment Canada, Chalk River, Ontario, K0J 1J0.

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[1] Lately, aforementioned form-class volume tables were formulated and converted into metric units by Berry (1981).

a step toward the accurate compilation of their volume." The methods of constructing the volume tables by means of taper tables have been demonstrated by many foresters, including Behre (1927), Heger (1965), Berry (1969) and Husch et al. (1972). Taper curves are useful tools (a) for the estimations of the number of roundwood products in trees of different sizes (Stiell and von Althen 1964), (b) for the computation of volume to meet any standard of utilization and any form of product (Belyea 1931) such as merchantable volume to a given top diameter, (c) for the prediction of any length of section at any location along the stem (Demaerschalk 1973), and (d) for the calculation of the length of any stem section per unit of volume (Kozak et al. 1968). Taper curves are also required by the harvesting equipment manufacturers who need to know at what height the merchantable diameter limit occurs, or what the stem diameter is at a certain height[2]. Such information is likewise needed for many other forestry purposes. Taper curves, along with wood density, can also be used in calculating the biomass of any given section on the stem. Finally, as suggested in the previously mentioned volume-tables publication (Anon. 1948), "...by use of these taper tables, entirely new form-class volume tables may be constructed to suit the requirements of local utilization."

Form class (FC), by definition, is "...any of the intervals into which a numerical expression of the taper may be divided for classification or use - commonly a range of form factors, form quotients or form-point heights" (Ford-Robertson 1971). Among these expressions of the taper, form quotient is the ratio of any diameter on a tree stem to the diameter at breast height (For instance, d_1/D or d_2/D, in Figure 1). If this ratio is expressed as being the diameter at half the tree height above breast height divided by the diameter at breast height, it is called absolute form-quotient (For instance, d at H/2 divided by D, in Figure 1). It is these absolute form-quotients which comprised the form classes used in constructing the taper tables in the original publication. These form classes are as 0.576-0.625, 0.626-0.675, 0.676-0.725 and 0.726-0.775, in terms of absolute form-quotients. The form classes were then expressed as 100 times the actual ratio of the mid-class, and were identified as 60, 65, 70 and 75.

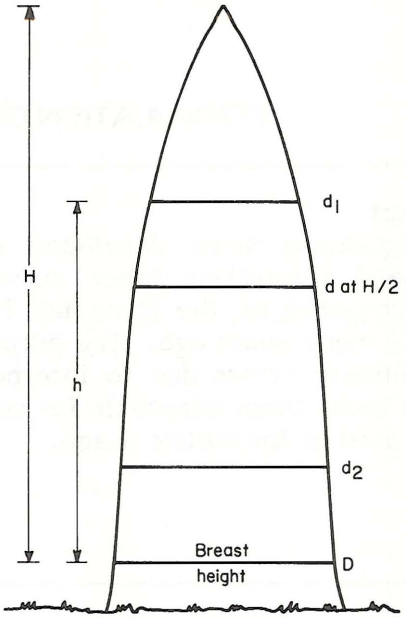

Figure 1. The illustration of the form-quotient concept.

The taper tables provided in the above mentioned form-class volume tables (Anon. 1930) enable us to estimate directly or by interpolation, for a specified form class, the relative diameter of a tree stem at a given relative height. In these tables, the relative diameter is defined as the ratio of a diameter inside bark on the stem (d) to the breast height diameter inside bark (D) measured at 4.5 feet (1.372 m), and the relative height is shown as the ratio of the height above 4.5 feet where diameter is measured (h) to the tree height above 4.5 feet (H). On a given tree by its D and H, once the relative height (h/H) is known, the stem diameter at this height will be found by multiplying the corresponding relative diameter (d/D) by the breast height diameter of that particular tree.

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[2] Newnham, R.M. 1971. A file report.

DATA

In order to formulate these curves - a total of 34 for eight tree species - the relative diameters are read on each curve at each relative height, to be treated later as the observed values (actually they are taken from the taper tables given in the 1948 edition of the above mentioned publication). In these tables, relative diameters were provided at each tenth part of the height above breast height, from 0.0 to 1.0. Relative diameters were shown as percentages. For each curve, 11 pairs of readings were available. These free-hand curves were originally based on a number of trees ranging from 243 (lodgepole pine) to 917 (the spruces).

METHOD AND ANALYSIS

Although these curves were drawn using the measurements taken in inches and feet, because their two variables are expressed in relative values, the ratios are dimensionless. Consequently, they are suitable for metric units in the present form. The only concern regarding metric practice would be the replacement of the location of breast height. The new breast height diameter is measured at 7.2 cm below the old location on the stem, and consequently the new height above breast height of the tree is 7.2 cm longer. When working with the observed relative diameter values in formulating the curves, this height difference will change the original relative-height values of the relative-diameter readings to new values of

$$\acute{p} = (p.H + 0.072)/(H + 0.072)$$

where $\acute{p}$ is the new relative height, p the original relative height and H as before, in metres (Figure 2). This change may require an adjustment on the observed data. However, because of the following reasons, no alteration regarding the height difference due to the new location of the breast height diameter in the metric system was found justified. When Figure 2 and Table 1 are studied together, it can be seen that:

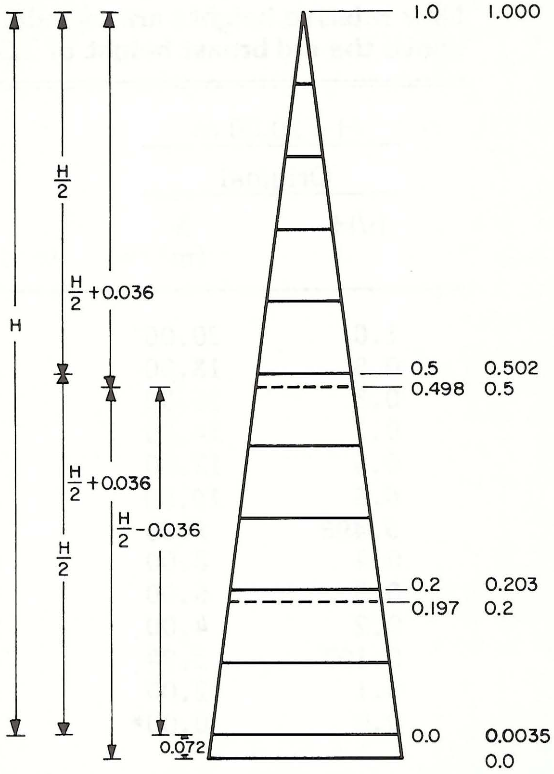

Figure 2. Diagram illustrating changes of locations and values of relative heights as a result of the change in new location of the breast height from 1.372 m to 1.300 m. New relative heights are calculated as an example for 20.00 m of height above the old breast height of 1.372 m.

First, the relative diameter 100.0% will no longer correspond to the relative height zero but to a value of around 0.0035, and the new relative height zero will carry a value of relative diameter slightly higher than 100.0%. These are neither acceptable nor are they possible to correct. Secondly, the difference between the old and the new locations of the same relative height is very insignificant and can be disregarded. For instance, in the given example, it is 3.6 cm at h/H = 0.5 and 1.4 cm at h/H = 0.2. Thirdly, d/D readings at the middle of the new height will not coincide with the original mid-values and therefore the new curves could no longer be called FC60, 65, 70 or 75. Fourthly, this new value of relative height corresponding to the original relative diameter is

Table 1. The old and the new relative heights and the heights at given locations on the stem as a result of the new breast height at 1.30 m. New relative heights are calculated as an example for 20.00 m of height above the old breast height of 1.372 m

H = 20.00 m		H = 20.072 m		
Original		New		Original
h/H	h (m)	h/H	h (m)	d/D (%)
1.0	20.00	1.000	20.072	0.0
0.9	18.00	0.900	18.065	13.4
0.8	16.00	0.801	16.078	26.1
0.7	14.00	0.701	14.070	38.0
0.6	12.00	0.601	12.063	49.3
0.5	*10.00*	*0.502*	*10.076*	*60.00*
0.498	*9.96*	*0.5*	*10.036*	*59.79*
0.4	8.00	0.402	8.069	70.5
0.3	6.00	0.302	6.062	79.4
0.2	4.00	0.203	4.075	86.9
0.197	3.94	0.2	4.014	-
0.1	2.00	0.103	2.067	93.3
0.0	0.00*	0.0036	0.072	100.0
		0.0	0.00 **	100.8

*At 1.372 m above ground level.
**At 1.300 m above ground level.

dependent on tree height, and the existence of H in this expression requires generating several curves for each of the present FC taper curves.

As a result of these arguments, the form-class indexes of 60, 65, 70 and 75 were not altered either.

When relative diameter values of the FC taper curves were plotted over relative height values, they formed a polynomial curve of a decreasing convex shape between h/H equal to zero and to 1.0. Some curves demonstrated a sigmoidal shape at the beginning of the curve, i.e., above the point of breast height. On every curve, when h/H = 0.0, d/D = 100.0%; when h/H = 1.0, d/D = 0.0%; and when h/H = 0.5, d/D = FC index. Because of this situation, whatever curve equation is chosen, it must be a conditioned equation.

These conditioned curves can be formulated in two ways: (1) by applying regression analysis to the 11 pairs of measurements, or (2) by using an algebraic method. If the equations are to be developed by the regression method, under

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If, for instance, this polynomial curve-equation is in the form of

$$Y = b_0 + b_1 \cdot X + b_2 \cdot X^2 + b_3 \cdot X^3$$

then, with the constraints governing the situation, that is, with

$b_0 = 100.0$, and

$b_1 + b_2 + b_3 = -100.0$

a regression analysis is to be done with its transformed model of

$$(Y - 100.0 + 100.0 \cdot X) = b_2 \cdot (X^2 - X) + b_3 \cdot (X^3 - X)$$

in order to establish the b_2 and b_3 parameters. Then, b_1 is to be determined by

$$b_1 = -100.0 - b_2 - b_3$$

knowing that this fit may not be the best. The values of the percent of variation explained and the standard error of estimate are to be calculated by using their own formulas with the observed data and these parameters.

the present circumstances there will be no difficulty in satisfying the first and the second conditions, i.e. having exactly the same d/D values for the two ends of the curves by using a transformation of the actual model if the model is chosen to be a polynomial curve (see footnote previous page). However, in this case, passing through the third point (FC value) would not be achieved easily. On the other hand, the algebraic method can satisfy any number of constraints, having any chosen powers of h/H. For this reason, the second method was followed for developing the conditional taper equations.

Because the objective is to have these conditioned curves resemble the original shapes of the free-hand curves as closely as possible, it was decided that the equations should pass through two more curve points in addition to the three conditions stated above. These were chosen at h/H = 0.2 (near the bottom of the stem) and at h/H = 0.7 (above the middle of stem). Another reason for this is that if only the first three points are used in developing the equations, regardless of tree species, all FC60 (or 65, 70, 75) curves would end up to be exactly the same. Furthermore, including the third and the fourth powers of the h/H variable in the equation would probably make it possible to fit the sigmoidal part of the curve.

Even though it was shown by Gray (1956) and Newnham (1965) and then confirmed by other researchers (Kozak and Smith 1966, Munro 1968) that a quadratic parabola and simple functions were well suited to description of bole shape, for the above reasons a function other than the quadratic had to be chosen. After these decisions were made, the following model was selected to formulate the curves:

$$Y = b_0 + b_1 \cdot X + b_2 \cdot X^2 + b_3 \cdot X^3 + b_4 \cdot X^4$$

where Y = d/D and X = h/H.

According to the constraints set above, b_0 is always 100.0 and the sum of the b coefficients is equal to –100.0. Therefore,

$$b_1 = -100.0 - b_2 - b_3 - b_4$$

When the coefficient b_1 in the model is replaced with this new b_1 and the model simplified, it becomes

$$Y = (1.0-X) \cdot 100.0 + (X^2-X) \cdot b_2 + (X^3-X) \cdot b_3 + (X^4-X) \cdot b_4$$

To be able to find the unknown coefficients of b_2, b_3 and b_4, we need three different equations with three pairs of d/D and h/H values from the observed data, such as indicated above (d/D at 0.2, 0.5 and 0.7). The coefficient b_1 will then be simple to calculate.

Following the establishment of the coefficients for each FC curve, one has to know how well the model has fit the data. This can be judged by the values of the percent of variation explained (R^2) and the standard error of estimate as percent of the mean (SEE%) of the predicted value. These can easily be calculated from their appropriate formulas and by using the observed data together with the established coefficients, in the above given model.

RESULTS

The equation coefficients for each of the 34 form-class taper curves were solved using the above model, and are provided in Table 2. This table also consists of the R^2 and the SEE% values. Extremely high R^2 and extremely low SEE% values indicate a near perfect fit. The good fit is to be expected because the equations were developed using observed data read off the already established curves, and the residuals for the five out of 11 observed values were set to zero. However, they also show how well the free-hand curves were originally drawn and how appropriately the chosen mathematical curve-model fits.

Using these established coefficients, all the curves were computer-plotted for every 0.01 value of X for checking purposes, and no peculiarities due to the third and the fourth powers of the independent variable were found.

Table 2. Coefficients of the form-class taper-curve equations

	Equation coefficients*					
Form Class	b_1	b_2	b_3	b_4	R^2	SEE%
			Balsam fir			
60	-54.612	-57.054	13.452	-1.786	0.9999	0.82
65	-62.322	45.655	-160.714	77.381	0.9998	0.96
70	-64.703	121.369	-291.190	134.524	0.9994	1.67
75	-54.642	107.976	-241.429	88.095	0.9995	1.39
			Jack pine			
60	-79.493	37.827	-97.024	38.690	0.9996	1.47
65	-85.089	76.756	-94.643	2.976	0.9997	1.12
70	-71.102	52.768	-40.595	-41.071	0.9994	1.63
			Lodgepole pine			
60	-72.173	-72.827	183.690	-138.690	0.9993	1.87
65	-93.542	81.875	-50.833	-37.500	0.9995	1.51
70	-62.023	-14.643	98.095	-121.429	0.9997	1.07
			Red pine			
			Over 120 yrs			
70	-92.797	96.131	-18.810	-84.524	0.9963	3.92
75	-85.953	129.826	-86.190	-57.143	0.9987	2.21
			Under 120 yrs			
70	-72.470	74.137	-95.119	-6.548	0.9967	3.91
75	-59.375	67.708	-87.500	-20.833	0.9999	0.61
			Eastern white pine			
			Over 120 yrs			
			Below 36.6 cm			
65	-97.917	101.250	-78.333	-25.000	0.9995	1.48
70	-71.905	85.238	-132.381	19.048	0.9999	0.33
75	-65.535	122.202	-207.857	51.190	0.9999	0.27
			From 36.6 cm to 49.5 cm			
65	-108.958	173.958	-219.167	54.167	0.9994	1.71
70	-103.542	218.542	-310.833	95.833	0.9999	0.77
75	-78.660	180.327	-290.357	88.690	0.9999	0.25

Table 2. (cont'd).

	Equation coefficients*					
Form Class	b_1	b_2	b_3	b_4	R^2	SEE%
			Above 49.5 cm			
65	-128.809	275.476	-384.762	138.095	0.9994	1.71
70	-106.993	225.327	-307.024	88.690	0.9996	1.27
75	-105.417	295.417	-448.333	158.333	0.9998	0.89
			Under 120 yrs			
60	-64.375	36.042	-117.500	45.833	0.9997	1.38
65	-68.185	123.185	-272.262	117.262	0.9999	0.77
70	-57.173	102.173	-206.310	61.310	0.9995	1.41
			Spruces			
			Below 24.1 cm			
60	-92.143	92.143	-171.429	71.429	0.9998	1.07
65	-56.190	-13.810	-25.238	-4.762	0.9999	0.57
70	-61.487	74.821	-174.048	60.714	0.9996	1.33
75	-49.970	54.970	-115.119	10.119	0.9999	0.67
			Above 24.0 cm			
60	-92.232	77.232	-126.071	41.071	0.9998	0.94
65	-93.215	136.548	-217.143	73.810	0.9997	1.18
70	-89.733	191.399	-326.071	124.405	0.9994	1.60
75	-63.006	128.006	-242.976	77.976	0.9993	1.66

*Coefficient b_0 is always 100.0.

However, it was noted that, wherever it occurs, the sigmoid shape of the curve was closely approximated.

When any of these equations are evaluated, it can be seen that for h/H = 0.0, d/D = 100.0; for h/H = 1.0, d/D = 0.0; and d/D values at the h/H values of 0.2, 0.5 and 0.7 agree with the original d/D values. The relative diameter at h/H = 0.5 is, of course, the FC name of the curve. An example of the original and estimated relative diameters or diameter ratios at different relative heights is provided in Table 3, using FC60 of balsam fir and FC75 of eastern white pine older than 120 years and larger than 49.5 cm of D. The curves of these formulated values are also shown in Figure 3.

As has been demonstrated to this point, three variables control the shape of the longitudinal profile of the stem: diameter at breast height, total tree height above breast height, and the absolute form-quotients represented by form classes. However, in a given tree species, changes in stem form with one of these variables is not easy to perceive, if the others are kept constant. This is illustrated in Figure 4 for a visual inspection of (a) trees of a specific diameter and total height, (b) trees of a specific form class and diameter, and (c) trees of a

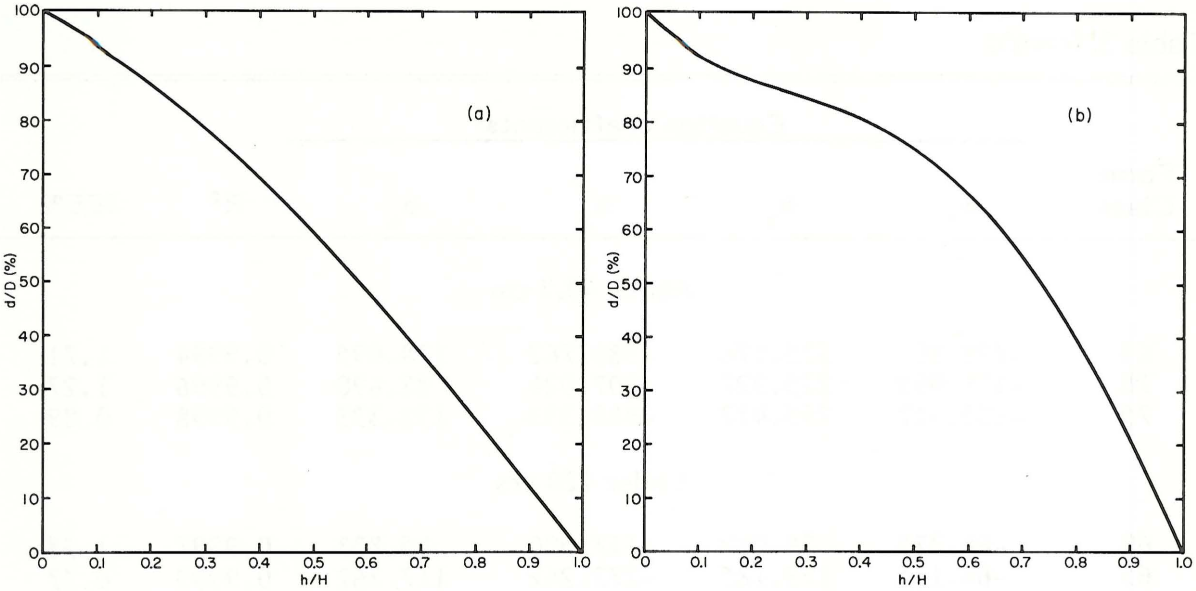

Figure 3. Formulated form-class taper curves of (a) FC60 of balsam fir, and (b) FC75 of eastern white pine older than 120 years and larger than 49.5 cm of D.

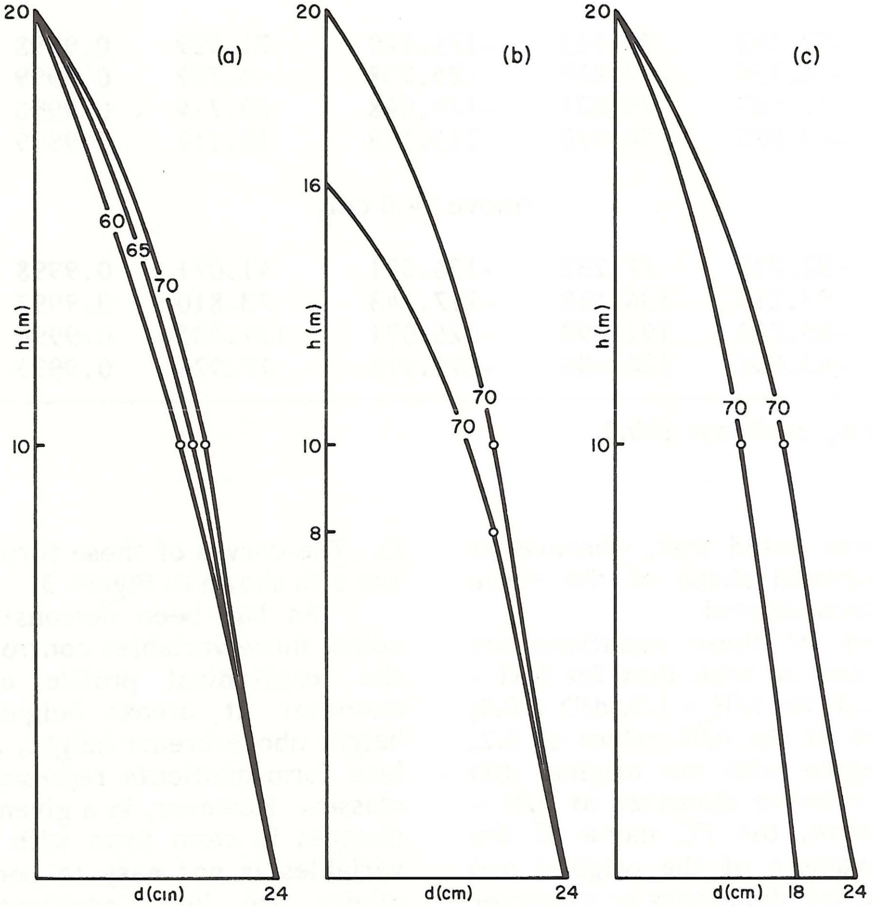

Figure 4. Stem forms in jack pine, above breast height, by the variables of breast height diameter, total tree height above breast height, and form class: (a) for a given diameter when heights are same, (b) for a given form class when diameters are same, and (c) for a given form class when heights are same.

Table 3. **Original and estimated diameter percentages for (a) FC60 of balsam fir, (b) FC75 of eastern white pine older than 120 years and larger than 49.5 cm of D**

	a		b	
h/H	Original d/D (%)	Estimated d/D (%)	Original d/D (%)	Estimated d/D (%)
0.0	100.0	100.0	100.0	100.0
0.1	93.3	94.0	91.0	92.0
0.2	86.9	86.9	87.4	87.4
0.3	79.4	78.8	83.8	84.1
0.4	70.5	69.8	80.0	80.5
0.5	60.0	60.0	75.0	75.0
0.6	49.3	49.4	67.0	66.8
0.7	38.0	38.0	55.2	55.2
0.8	26.1	26.0	40.6	40.0
0.9	13.4	13.3	22.0	21.5
1.0	0.0	0.0	0.0	0.0

specific form class and total height.

One of the advantages of these curve formulas, as mentioned earlier, is to be able to estimate diameters at any point along the stem without interpolating between the tenths of the height when handling large numbers of trees.

These developed equations are good for metric units and for breast height diameters taken at 1.30 m above ground.

ACKNOWLEDGMENT

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