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A variable-form taper function

R.M. Newnham

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Petawawa National Forestry Institute
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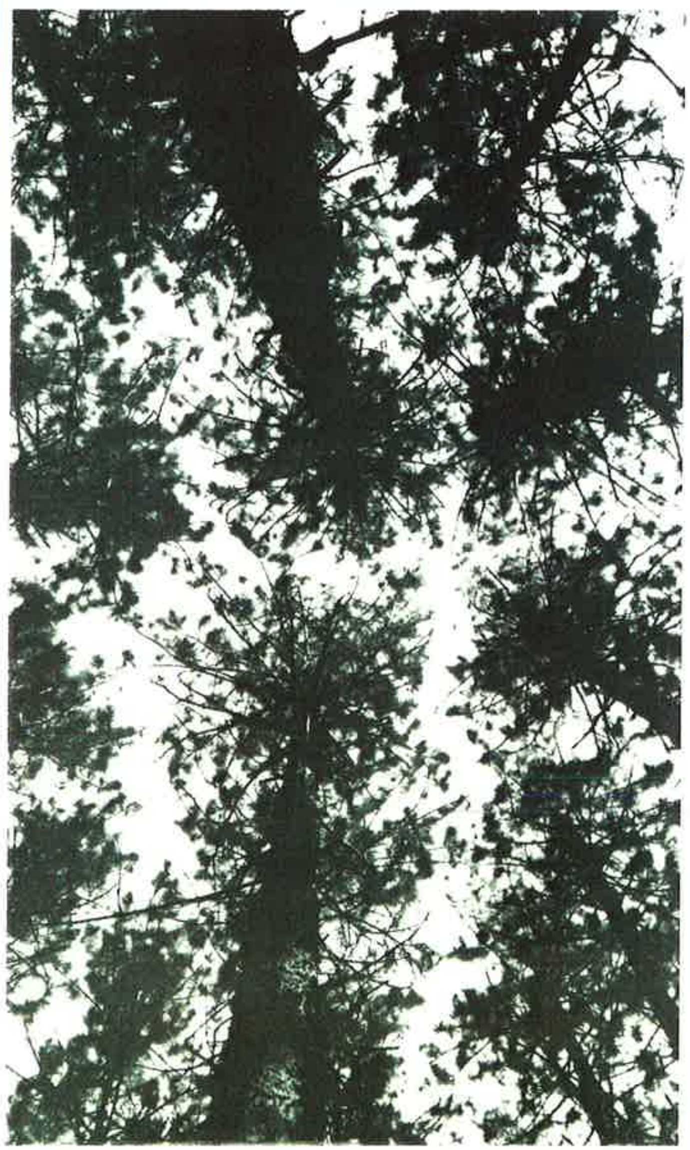

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A VARIABLE-FORM TAPER FUNCTION

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ABSTRACT

The theories that have been developed to account for stem form are briefly described, and published taper functions reviewed. A new taper function for red pine (*Pinus resinosa* Ait.) plantation trees is developed that allows for continuous variation in form along the stem. Form is expressed by the variable k in the relationship:

$$(d/D)^k = (H - h)/(H - 1.30)$$

where D is the diameter at breast height (1.30 m), H the total height of the tree, and d the diameter at a height h on the stem. From stem analyses, estimates of k can be obtained at any point from $\ln[(H - h)/(H - 1.30)]/\ln(d/D)$. The relationship between k and relative height and D/H ratio is then calculated using standard regression methods. The two models that appear to be the most promising are:

$$k = 2.48 - 1.540X^6 - 0.696(D/H) + 0.770X^2.(D/H)$$

and $k = 2.58 - 0.763(D/H) + 0.205X.(D/H)^2 - 0.244 (1/h)$

where $X = (H - h)/(H - 1.30)$. These models give accurate and relatively unbiased estimates of stem diameters that compare favourably with the estimates of the more complex Max-Burkhart segmented polynomial function. The two variable-form models give more accurate and less biased estimates of stem volume than those obtained using the Max-Burkhart model.

RÉSUMÉ

Les théories qui ont été élaborées pour tenir compte de la forme de la tige sont brièvement décrites, et l'on passe en revue les fonctions de mesure du défilement qu'on peut trouver dans la documentation publiée. Une nouvelle fonction, valable pour des arbres de plantations de pins rouges (*Pinus resinosa* Ait.) est présentée; cette fonction tient compte d'une variation continuelle de forme le long de la tige. La forme est exprimée par la variable k dans l'équation suivante :

$$(d/D)^k = (H - h)/(H - 1{,}30)$$

où D est le diamètre à hauteur de poitrine (1,30 m), H est la hauteur totale de l'arbre et d est le diamètre à une hauteur h, le long de la tige. Par des analyses de la tige, on peut obtenir des estimations de k à n'importe quel point en exprimant l'équation comme suit : $k = \ln[(H - h)/(H - 1{,}30)]/\ln(d/D)$. La relation entre k et la hauteur relative et le rapport D/H est alors calculée au moyen de méthodes de régression normales. Les deux modèles qui semblent les plus prometteurs sont les suivants :

$$k = 2{,}48 - 1{,}540X^6 - 0{,}696(D/H) + 0{,}770X^2.(D/H)$$

et $k = 2{,}58 - 0{,}763(D/H) + 0{,}205X.(D/H)^2 - 0{,}244 (1/h)$

où $X = (H - h)/(H - 1{,}30)$. Ces modèles fournissent des estimations précises et relativement non biaisées des diamètres de la tige, lesquelles se comparent avantageusement avec les estimations obtenues au moyen de la méthode plus complexe faisant appel à la fonction polynômiale segmentée de Max-Burkhart. Les deux modèles de défilement de forme variable donnent des estimations plus exactes et moins biaisées du volume de la tige que celles obtenues avec le modèle de Max-Burkhart.

A VARIABLE-FORM TAPER FUNCTION

INTRODUCTION

The topic of stem form and taper has been studied by foresters for over one hundred years and, from a survey of recent literature, still appears to be a subject of high priority in forest research. There are perhaps two reasons for this. First, no single theory has been developed that adequately explains how stems vary in form, both within and among trees. Thus, it has not been possible to develop a satisfactory taper function that would be uniformly acceptable over a wide range of conditions. Second, and more important from a practical point of view, a taper function that can accurately predict the diameter at any point on the stem from one or two readily measured variables is essential for estimating the volume of standing trees and the construction of volume tables to different merchantable limits. Such equations will also be useful for estimating the distribution of log sizes by top diameter and length, information that is needed for planning better manufacturing facilities. Kilkki and Lappi (1987) note that taper functions have increasingly replaced ordinary volume tables for these purposes. Now that wood shortages are occurring in Canada, and a greater proportion of standing timber is being harvested and converted to high-value products or used to produce energy, the need for accurate volume estimation for planning purposes is very great. Also, as the country moves into more intensive forest management and begins to use wood from man-made forests, existing volume functions and tables that were based on information from unmanaged natural stands may no longer be appropriate (see, for example, Amateis and Burkhart 1988a).

Four main theories have been developed to account for the form of forest trees. These have been described in some detail and discussed by Larson (1963). The proponents of each of the theories --- nutritional, water conduction, mechanistic, and hormonal --- have each been able to demonstrate that their particular theory is applicable. This indicates that, either each may be applicable under certain conditions or, more probably, there are elements of each theory that hold true. However, all theories agree on certain points:

(1) Above the region of butt swell, the greatest taper occurs in that portion of the stem within the live crown. Both ring width and ring area increase with increase in distance from the top of the stem, indicating that the stem there is probably conical (or even neiloidal) in form.

(2) The maximum growth in ring area occurs near the base of the crown, and the minimum at some point between the butt-swell maximum and the base of the crown. Both the minimum and maximum move upwards in dry years and downwards in excessively wet years.

(3) Below the live crown, the rate of growth is largely governed by the position of the tree within the crown canopy. For free-growing trees, the ring area may continue to increase down the stem. For trees in the upper canopy, ring area may remain constant so that ring width will consequently decrease. Both ring width and area decline down the stem for suppressed trees.

(4) Butt swell is very variable but appears to have a support function.

(5) The crown, particularly crown length (although differences in crown length may be offset by differences in crown width, crown density, and needle persistence to some extent), plays a decisive role in determining stem form.

Larson (1963) did not feel that the more recently developed hormonal theory supplants earlier theories but, rather, provides the physiological basis for them. Kozlowski (1971) stated that the formation of wood along the stem is governed more by the physiology of the tree than its strength requirements --- the fact that the stem is also mechanically efficient may be fortuitous. However, under certain conditions, tree stems are known to respond to stress with, for example, the formation of compression or tension wood.

Heger (1965) and Smith (1980) separated radial growth along the stem into "earlywood" and "latewood". Heger postulated that temperature was an important factor governing radial growth and that differences in growth along the stem may be due to air temperature gradients. The different shapes of the earlywood and latewood layers reflect the respective spring and summer environmental energy gradients. Smith found that, for Douglas-fir (*Pseudotsuga menziesii* [Mirb.] Franco), maximum growth of earlywood and latewood occurred near the base of the full crown.

Most theories describe stem form in qualitative terms. It is only the mechanistic theory, largely developed by Metzger (Busgen and Munch 1929) and subsequently modified by Gray (1956), that attempts to develop a functional relationship between stem diameter and height. Because he postulated that wood formation in the stem was governed by its requirement for strength, Metzger described the stem as a beam of uniform resistance to bending (particularly to forces brought about by wind), with one end fastened in the soil. Such a beam would have the form of a cubic paraboloid. He was able to show that, below the centre of gravity of the crown, diameter cubed plotted on height was more or less a straight line. Gray (1956) claimed that, as the stem was not held rigidly at its base, the cubic paraboloidal form represented an overexpenditure of material for the strength requirements of the stem. A quadratic paraboloidal form, in which diameter squared was linearly correlated with height, would be more efficient. Newnham (1965) found that this relationship held well for that portion of the stem between 15 and 80 per cent of the total height and used it to study the variation in taper with age and thinning regime in coniferous species. It should be noted that two of the most commonly used formulae for calculating log volumes (those of Smalian and Huber) assume that the stem has the form of a quadratic paraboloid.

DEFINITIONS

For tree stems, the terms "taper" and "form" are often used interchangeably in the literature. However, in this report they will have the following specific meanings:

Taper. The rate of decrease in diameter with increase in height up the stem.

Form. The geometric shape of the tree stem. Tree stems have often been considered to be comprised of three sections: a conical top section, a paraboloidal section below the live crown, and a neiloidal butt section. In this report, stem form is considered to vary continuously along the stem and is expressed by the variable k. The following definitions for variables are used through the remainder of this report:

Y = relative diameter d/D
X = relative height, $(H - h)/(H - 1.30)$
$Z = h/H$
D = diameter at breast height -- DBH (1.30 m)
H = total height
h = height above ground level
d = diameter at height h
b_i = constants

Diameters may be measured either inside or outside bark -- outside bark measurements are used for the analyses of the black spruce and red pine data that are described later in this report. Other variables are defined as required.

REVIEW OF EXISTING TAPER FUNCTIONS

While physiologists have been attempting to discover a satisfactory theory for stem form, mensurationists have striven to develop mathematical functions that would describe the profile of the stem from the ground to the tip. Early efforts produced relatively simple formulae, e.g. that of Hojer (Husch 1963):

$$Y = b_0 \ln[(b_1 + X)/b_2]$$

A commonly used formula in North America was that of Behre (1923):

$$Y = X/(b_0 + b_1X)$$

These early formulae gave satisfactory fits to stem profiles over most of the merchantable portion of the stem except for the region of butt swell. Behre "adjusted" diameters in this region to obtain a better fit. The formula is still used occasionally (see Wiant and Charlton 1984, and Ormerod 1986). Graphical methods were used by Stiell

(1960) and Stiell and von Althen (1964) to develop taper curves for plantation-grown red pine (*Pinus resinosa* Ait.).

Kozak et al. (1969) used a quadratic polynomial

$$(d/D)^2 = b_0 + b_1Z + b_2Z^2$$

subject to the restriction $b_0 + b_1 + b_2 = 0$.

In this case, d was measured inside bark and D was the stump diameter inside bark (the DBH outside bark was used to estimate this). For some of the species on which the model was tested, negative estimates of upper stem diameter were obtained. For those species (coastal spruce and cedar in British Columbia), a conditioned function could be used:

$$(d/D)^2 = b_1 (1 - 2Z + Z^2)$$

The Alberta Forest Service (1987) more recently tested the unconditioned Kozak et al. function in a different form:

$$(d/D)^2 = b_1 (Z - 1) + b_2 (Z^2 - 1)$$

Alemdag (1983) used a constrained quartic polynomial to formulate the hand-drawn, form-class taper curves for eastern Canadian commercial tree species.

Desirable features of any taper function are that it should be possible to directly estimate height for any stem diameter (useful for determining merchantable height to a given diameter limit), and that the taper function should be integratable to form a compatible volume function. If either of these conditions do not exist, time-consuming iterative procedures have to be used. Munro and Demaerschalk (1974) have discussed the advantages of compatible volume and taper functions. Although the usual approach is to develop the taper function first and then the volume function, Demaerschalk (1973), Amateis and Burkhart (1988b), and Alemdag (1988) have proceeded in the opposite direction by deriving taper functions from existing volume functions.

With the advent of the computer in forest research in the early 1960s, more sophisticated methods were used for deriving taper functions. Fries (1965) used principal component analysis (PCA) to study the form of birch and pine trees in Sweden and British Columbia. Kozak and Smith used the same method to define taper in several commercial tree species in British Columbia, and Liu and Keister (1978) did likewise with loblolly and slash pines (*Pinus taeda* L. and *P. elliottii* Engelm.). In each case, the first eigenvalue (component) accounted for more than 99 per cent of the variance. One disadvantage of this method however, is that all diameters have to be measured at fixed percentages along the stem.

The "whole-bole" system was developed by Demaerschalk and Kozak (1977), who divided the stem into two sections at the inflection point (the point where the form of the stem changes from neiloidal butt swell to the parabolic form of the upper stem). The inflection point, which was determined by eye after plotting relative diameter, d/D, on relative height, (H - h)/H, was found to vary between 20 and 25 per cent above ground level for 32 species groups in British Columbia. The diameter D_i at this inflection point (rather than at breast height) was used as the base for the diameter ratios. The equation for the top portion of the stem was:

$$d/D_i = (Z/R_I)^{b_1} . b_2^{(1-Z/R_I)}$$

and for the bottom portion:

$$d/D_i = b_3 - (b_3 - 1) [(1-Z)/R_I]^{b_4}$$

where R_I is the relative distance of the inflection point from the top of the tree. The b_i are conditioned to ensure that the expected diameters coincide with the observed diameters at the top, at the inflection point and at breast height, and that the transition from one curve to the other is smooth. The results of the authors' tests showed that this dual equation was remarkably precise and accurate. The disadvantages of this method are that solution of the bottom equation requires sophisticated software, and the equation for the top part cannot be directly integrated to form a volume function. D_i is not measured directly but has to be estimated from D (using a second-degree polynomial equation) and, because it assumed that the inflection point was located above breast height, this taper system cannot be used for small trees. Besides the British Columbia Ministry of Forests, it has been applied in that province by Layden (1984).

Goulding and Murray (1976) used a fifth degree polynomial and Bruce et al. (1968) used a regression with six terms (including values of X raised to the power 40). Liu (1980) used cubic spline functions to portray stem taper in yellow poplar (*Liriodendron tulipifera* L.). This method divides the stem into a number of sections and a cubic polynomial is fitted through the data points within each section. Constraints are imposed on the first and second derivatives to ensure a smooth transition at each join point. The number of sections and the length of each section are selected to give the best fit. Biologically, it is difficult to justify functions of this complexity.

Max and Burkhart (1976) used a segmented polynomial regression to develop taper equations for loblolly pine natural stands and plantations. This method could be considered intermediate in complexity between the whole-bole system of Demaerschalk and Kozak and the spline method of Liu. The stem is divided into only three sections and separate conditioned polynomial equations calculated for each section. However, the location of the join points is selected by the model to give the best fit to the stem profile. Max and Burkhart found that the most satisfactory equation system was a quadratic-quadratic-quadratic model of the form:

$$(d/D)^2 = b_1 (Z - 1) + b_2 (Z^2 - 1) + b_3 (a_1 - Z)^2 I_1 + b_4 (a_2 - Z)^2 I_2$$

where a_1 and a_2 are the relative distance from the top of the tree of the upper and lower join points respectively, and:

$$I_1 = \begin{cases} 1, & 0 < Z \leq a_1 \\ 0, & a_1 < Z < 1 \end{cases}$$
$$I_2 = \begin{cases} 1, & 0 < Z \leq a_2 \\ 0, & a_2 < Z < 1 \end{cases}$$

Most statistical software packages have procedures for calculating this function, although a considerable amount of computing time and expense may be required. Another advantage is that it can be directly integrated to give a volume function. In its present form there is no guarantee that predicted and observed diameters at breast height will be the same. This could be corrected by substituting X for Z in the formula. The Alberta Forest Service (1987) found that, of 15 functions that were tested, the Max-Burkhart was the best for general application in that province. Cao et al. (1980) examined several taper functions and also found that this model was the best for estimating diameters along the stem, but found inconsistencies when it was integrated to give volume. A segmented taper equation, based on that of Goulding and Murray (1976), proved to be the best all-round model for taper and volume estimation.

Two nonlinear regression functions that are of particular interest, because they relate to the new taper function that will be described later in this report, are those of Ormerod (1973) and Forslund (1982). They both recognized that the form of a stem may be other than that of a cone, paraboloid, or neiloid. Ormerod's basic function was:

$$d/D = [(H - h)/(H - k)]^p$$

where D is the measured diameter at height k and $p > 0$. The inflection point k was considered to be fixed at 30 per cent of the total height and the function was tested on a data set similar to that used by Kozak et al. (1969). The values of p ranged between 1.0 and 4.8 for the upper stem and between 0.57 and 0.89 for the lower section.

Forslund's model was similar:

$$d/D = [1 - (h/H)^{b_1}]^{1/b_2}$$

He considered only the case where $b_1 = 1$ which resulted in $b_2 = 1.5$, a form intermediate between a cone and a paraboloid that he called a "paracone".

In a later model, Ormerod (1986) assumed that there were two join points (at 0.2 or 0.25 and at 0.65 of total height). A complicated method, based on the Behre formula, was developed for describing the stem profile in each section.

A number of other complex taper functions have been developed, some of which were examined by Byrne and Reed (1986).

Recently, Sweda (1984) has proposed deriving taper functions from theoretical growth functions for height and diameter. The development of such integrated functions is appealing, although there appear to be some practical difficulties to overcome.

Grosenbaugh (1966) observed that tree stems assume an infinite number of shapes and that it is difficult to develop a single, simple, accurate equation to describe the taper of the stem. Each stem has a number of inflection points and "... the traditional conoid, paraboloid, and neiloid are merely convenient instances in a continuum of short monotonic shapes." While several authors have recognized that other geometric forms are possible, no one has recognized that the form of a stem does not change abruptly from one geometric form to another. The purpose of the present study was (a) to demonstrate that stem form changes continuously with height and (b) to develop a function that would satisfactorily describe those changes.

METHOD

As has been noted in the previous section, the form of a tree stem has often been considered to consist of three components: a conical section within the live crown, a paraboloidal section for the main portion of the stem, and a neiloidal butt section (Figure 1). The general formula describing the profile of each of these sections is:

$$d^k = b_0 + b_1 (H\text{-}h) \quad [1]$$

or, in proportional terms:

$$Y^k = b_0 + b_1 X \quad [2]$$

where X and Y are as previously defined. The value of k would be 1 for a cone, 2 for a quadratic paraboloid, 3 for a cubic paraboloid, and 2/3 for a neiloid. This relationship between k and height is illustrated in Figure 2. However, the hypothesis in the present study is that k does not change abruptly (the "steps" in Figure 2) at the join points of the different geometric forms but, rather, changes gradually as illustrated by the curve. Thus k would be 1 near the top of the stem, gradually increasing to a value between 1 and 2 near the base of the crown. Moving further down the stem, k would continue to increase (perhaps to a value of 3 or more) until the influence of butt swell came into effect. There would then be a rapid decrease until a final value for k of 2/3, or less, would be reached. The parameter k is no longer a constant (at least within the prescribed stem sections) but a variable that is related to height (or distance (H-h) from the top of the stem).

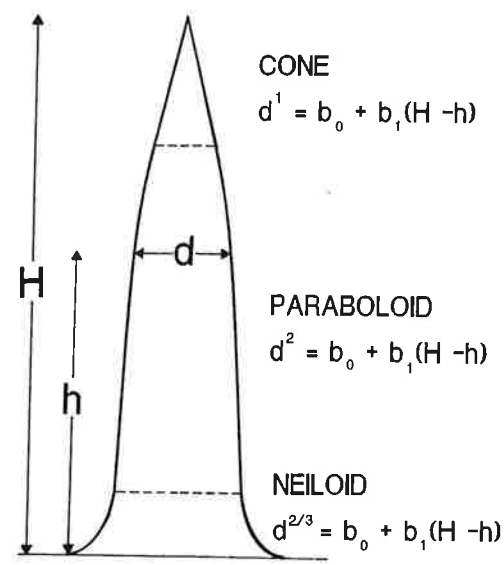

Figure 1. The conventional geometric forms for the three sections of a tree stem.

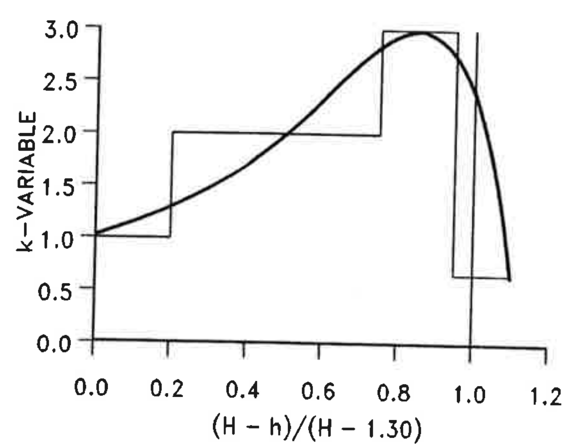

Figure 2. The theoretical relationship between the form parameter, k, and relative height.

To develop the relationship between k and (H-h), it is necessary to obtain estimates of the value of k from measurements of diameter at different heights along the stem. The value of k at any point i (except for the first and last measurements) can be estimated from the points immediately below and above it using the formula:

[3] $$Y_i^k = Y_{i-1}^k + (Y_{i+1}^k - Y_{i-1}^k).(X_i - X_{i-1})/X_{i+1} - X_{i-1}$$

The calculation has to be done by iteration and so can be time consuming and costly if a large number of trees are involved.

The second method of estimating values of k is much simpler. If k is considered to be a continuous variable, the basic form equation [2] must be constrained so that Y = 0 when X = 0 (top of the tree) and Y = 1 when X = 1 (breast height). This can only happen when $b_0 = 0$ and $b_1 = 1$ so that [2] becomes:

[4] $$Y^k = X$$

If the kth root of both sides of the equation is taken, [4] can be seen to be similar to the basic functions used by Ormerod (1973) and Forslund (1982):

[4A] $$Y = X^{1/k}$$

Taking logarithms of both sides of [4], and transposing, gives:

[5] $$k = \ln(X)/\ln(Y)$$
$$= \ln[(H - h)/(H - 1.30)]/\ln(d/D)$$

Thus, estimates of k can be obtained at any point on the stem where height and diameter are known --- except at breast height where d = D (because ln(1) = 0). Problems may also expect to be encountered with measurements that are very close to breast height where, as d -> D, ln(d/D) -> 0 and even small measurement errors or stem irregularities could result in high and unrealistic values for k. This should not be a major problem in practice as measurements are seldom taken at less than 50 cm from breast height, so that only the breast height measurements would have to be discarded.

A number of linear and nonlinear regression models were tested to establish the relationship between k and X. These will be described in detail in the results section.

DATA

For testing different taper functions, two sets of data were used.

Black Spruce

This data set consisted of 15 black spruce (*Picea mariana* [Mill.] B.S.P.) trees that had been felled in the Petawawa Research Forest, Chalk River, Ontario. For each tree, DBH and total height were recorded and diameter (outside bark) recorded at stump height 0.25 ft (= 0.075 m), 0.50, 0.75, 1.00, 1.5, 2.0, 2.5, and 3.5 ft. Above breast height (4.5 ft or 1.37 m), diameter measurements were taken at 2-ft (0.60 m) intervals to the base of the live crown. Thereafter, measurements were taken at the mid-point of each internode. The trees were thus intensively sampled, with the number of measurement points per tree being as high as 36. The original measurements were converted to SI units and DBH at 4.5 ft converted to DBH at 1.30 m using the formula given by Alemdag and Honer (1977). DBH ranged between 11.3 and 28.04 cm, with a mean of 16.3 cm, and total height between 11.4 and 19.2 m, with a mean of 14.6 m. No information is available about the stand in which the trees were growing.

Red Pine

The second set of data was obtained from permanent sample plots in the red pine spacing trials that were established in 1953 near Chalk River. Spacings ranged between 4x4 ft (1.2 x 1.2 m) and 14x14 ft (4.3 x 4.3 m), there being two plots at each spacing (except at 10 ft where there were three). These plots were measured in 1962 and at five-year intervals thereafter, the latest measurement being in 1987. Besides the usual measurements of DBH, total height, crown length, and crown width, diameters outside bark were also recorded at stump height (0.5 ft) and, by climbing the trees, at 5-ft (1.52 m) intervals above breast height to the top of the tree (it appears that, in the most recent measurements, some of the higher measurements were missed). Diameter at the mid-point on the stem was also recorded. As with the black spruce data, the original measurement units were converted to SI units.

The data for all spacings and years were pooled and trees with less than seven diameter measurements (not including the midpoint measurement) above breast height were discarded. This left a total of 548 trees (including trees that were measured more than once), mostly from the last three plot measurements. The data set was then divided into two subsets. The first, for use in the analysis, was obtained by selecting at random one sample plot from each of the spacings. The remaining subset was used as a control. Statistics for the two subsets are given in Table 1.

three adjacent diameter measurements on the stem that should be fairly close together. The intensively sampled black spruce data appeared to be ideal for this purpose. Unfortunately, besides being time consuming, the results were very disappointing. There were extreme fluctuations in the values of k and, even after eliminating the most extreme values and smoothing the remainder, there was still considerable variation (see, for example, tree 4 in Figure 3). This method was therefore abandoned.

Figures 4A and 4B show the values of k obtained using equation [5] for black spruce trees 4 and 13. It can be seen that, apart from one or two "outliers", the variation in k values has been reduced considerably. For tree 4 that had a live crown ratio of 24 per cent and was therefore probably in the intermediate or suppressed crown class, k was approximately 1.5 near the top of the stem and then increased to a maximum of about 2.5 above breast height. It then decreased rapidly to a value of about 0.5 at the stump. By contrast, tree 13 was probably relatively free-growing as it had a live crown ratio of 89 per cent. The value of k remained fairly constant at about 1.3 from the top to above breast height and then declined to a little above 0.5 at the stump. Three-term polynomial regressions were calculated for the relationship between k and height for each as indicated in Figure 4. The taper curves for the two trees

Table 1. Summary statistics for the red pine plantation data

Variable	N	Mean	Standard deviation	Range	
				Minimum	Maximum
Working data:					
DBH (cm)	272	24.51	6.77	11.15	42.14
Total Height (m)	272	16.43	1.58	13.17	19.51
Live Crown Ratio (%)	272	45.87	11.29	20.21	76.61
DBH/Total Height Ratio	272	1.50	0.42	0.72	2.47
k-parameter	2554	1.53	0.56	0.17	3.78
Control Data:					
DBH (cm)	276	22.44	5.79	9.23	36.23
Total Height (m)	276	16.89	1.84	13.29	20.73
Live Crown Ratio (%)	276	41.95	7.93	27.35	73.16
DBH/Total Height Ratio	276	1.34	0.37	0.62	2.18
k-parameter	2613	1.57	0.64	0.12	7.30

The range of initial spacings in the red pine data ensured that there would be a wide range in form among the individual trees. At the widest spacing (14x14 ft), the crown canopy was still fairly open in 1987 and the trees thus subjected to a minimum of competition.

RESULTS

k-Parameter Values

As was described in the methods section, two methods of obtaining values of k from the stem measurement data were tried. The first required

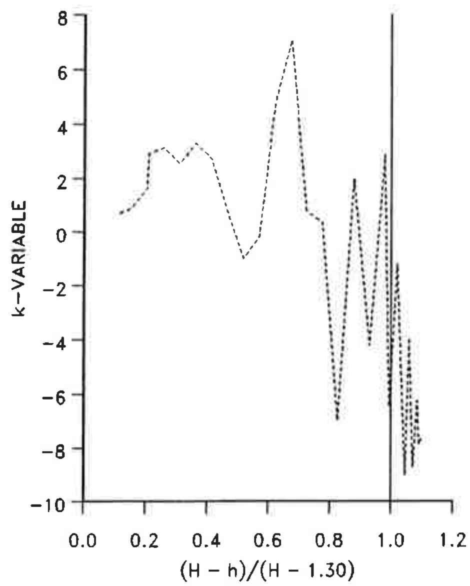

Figure 3. An example of the values of k obtained by iteration for black spruce tree 10.

(Figures 4C and 4D) were then drawn by substituting values for k in the equation:

[6] $d = D[(H - h)/(H - 1.30)]^{1/k}$

These results, and those obtained from the other black spruce trees, appear to comply with the hypothesis presented earlier. However, even with fairly free-growing trees, the upper part of the stem apparently does not become conical in form but is intermediate between a cone and a quadratic paraboloid (the "paracone" of Forslund (1982)). It can also be seen that the taper curves obtained from [6] give very good fits to the stem over its entirety.

The black spruce data set, although ideal for exploratory testing, was not sufficiently large for more comprehensive evaluation of the taper functions that were to be investigated. For this, the red pine plantation data were used. Values of k were again obtained using [5]. With such a large data set, it was not possible to screen outliers by eye. Instead, the nine measurements that gave a value of k > 4 were discarded, as were the 272 measurements at breast height (for reasons given previously). For the control data set, only the 276 breast height measurements were discarded, the twelve measurements with values of k > 4 were retained. Statistics for the k-parameter are given in Table 1.

In Figure 5, values of k have been plotted over relative height for the working data set. Although there is a considerable scattering of points, the general trend appears to be similar to that of the two spruce trees in Figure 4. Trees with small crowns tend to have higher values of k for the portion of the stem above breast height than do more free-growing trees. Figure 5 also illustrates a rather large gap in this data set around a value of 1.0 for (H - h)/H - 1.30) because of the necessity to discard the breast height data. For this, and presumably other taper studies, additional stem diameter measurements should be taken at 0.8 m and also, if possible, at 1.8 m, so that the butt-swell portion of the stem can be modelled more accurately.

Regression of k on Relative Height

Twenty-four regression models were tested to describe the relationship between the k-parameter and relative height. Linear regression analyses were performed using the SAS STEPWISE procedure (SAS Institute Inc. 1985). The MAXR option was used whereby, at each step, several models are tested in an attempt to find the regression model with the highest coefficient of determination, R^2. To keep the functions simple, it was decided that no more than three independent variables would be included. Addition of a fourth variable did not generally produce a significant improvement and, although the contribution of the third variable was sometimes not significant, it was required to give a better fit in the butt-swell portion of the stem. Nonlinear regression analyses were done using the secant (DUD) computational method of the SAS NLIN procedure (*ibid.*).

Only four of the 24 regression models are described here but the remainder (Models 6 - 24) are listed in Appendix I. These four are examples of the different forms that were tested and, in general, they gave the best fits to the basic data (the working data set of 272 red pine trees). For comparative purposes, the Max-Burkhart segmented polynomial regression was also calculated.

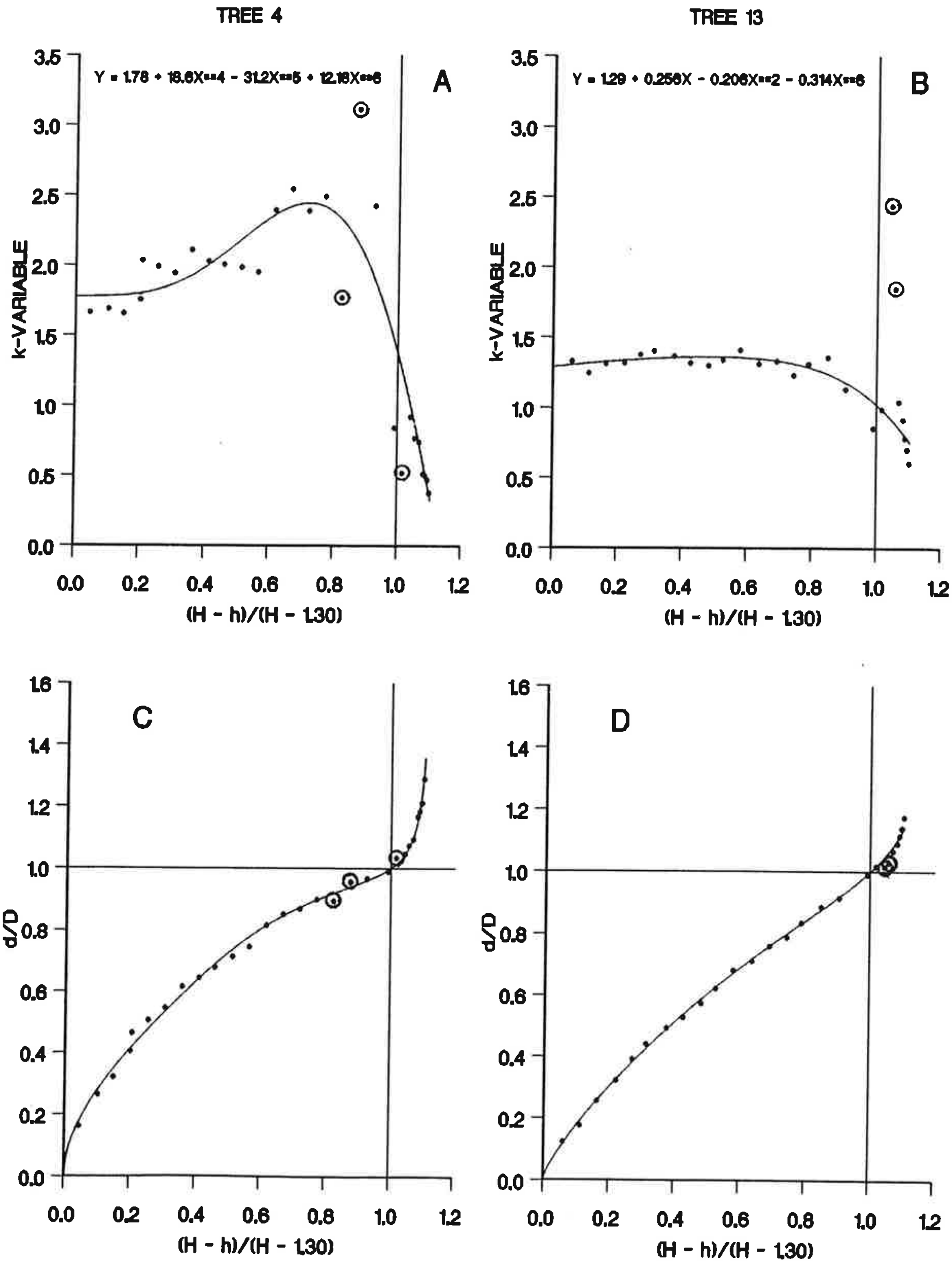

Figure 4. The relationship between k and relative height for black spruce trees 4 (live crown ratio = 24%) and 13 (live crown ratio = 89%), and the resultant taper curves. Circled observations were omitted from the regression calculations.

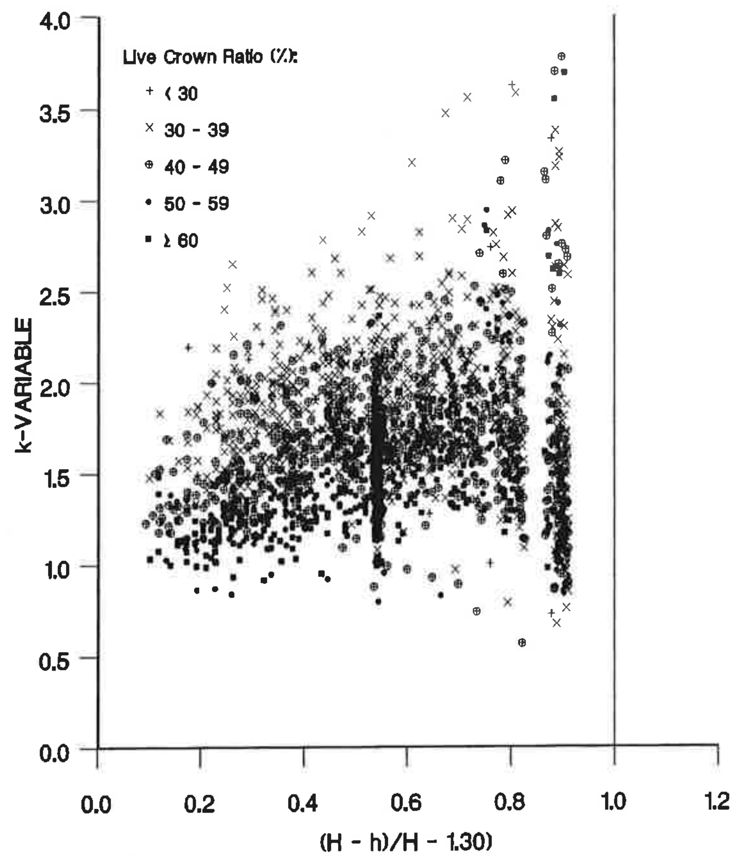

Figure 5. The relationship between k and relative height for the 272 red pine trees in the working data set.

Model 1. The relationship between k and relative height, X, was treated as a polynomial regression in which values of X up to the sixth power could be included. The final three-term regression was:

[7] $$k = 1.28 + 0.772X + 2.234X^5 - 3.198X^6$$
$$R^2 = 0.5654$$

The inclusion of the third variable caused only a 0.1 per cent increase in the value of R^2. A three-term polynomial, in which the independent variable was $W = \exp(X) - 1$, gave a slightly higher R^2 (0.5677 -- see Model 8 in Appendix I). Model 1 is illustrated in Figure 6A, together with the resulting taper curve (Figure 6B) obtained by substituting [7] for k in [6].

Model 2. For this model, separate simple linear regressions were calculated for the upper and lower portions of the stem. The join point was determined by iteration, the final selection being that point that minimized the sum of the squared deviations from the combined regression lines. The final model was:

[8] $$k = \begin{cases} 1.83 + 0.526(X - 0.836) & \text{for } X \leq 0.836 \\ 1.83 - 6.133\,(X - 0.836) & \text{for } X > 0.836 \end{cases}$$

The sum of the squared deviations $\sum(k - \hat{k})^2$, was 349.30. An equally good fit was obtained when X^2 was used as the independent variable but there was a poorer fit with W (Models 13 and 6 in Appendix 1). Although the abrupt change in the relationship between k and relative height at the join point (Figure 6C) is nowhere as noticeable in the corresponding taper curve (Figure 6D), it is still undesirable (although, possibly, of minor practical importance).

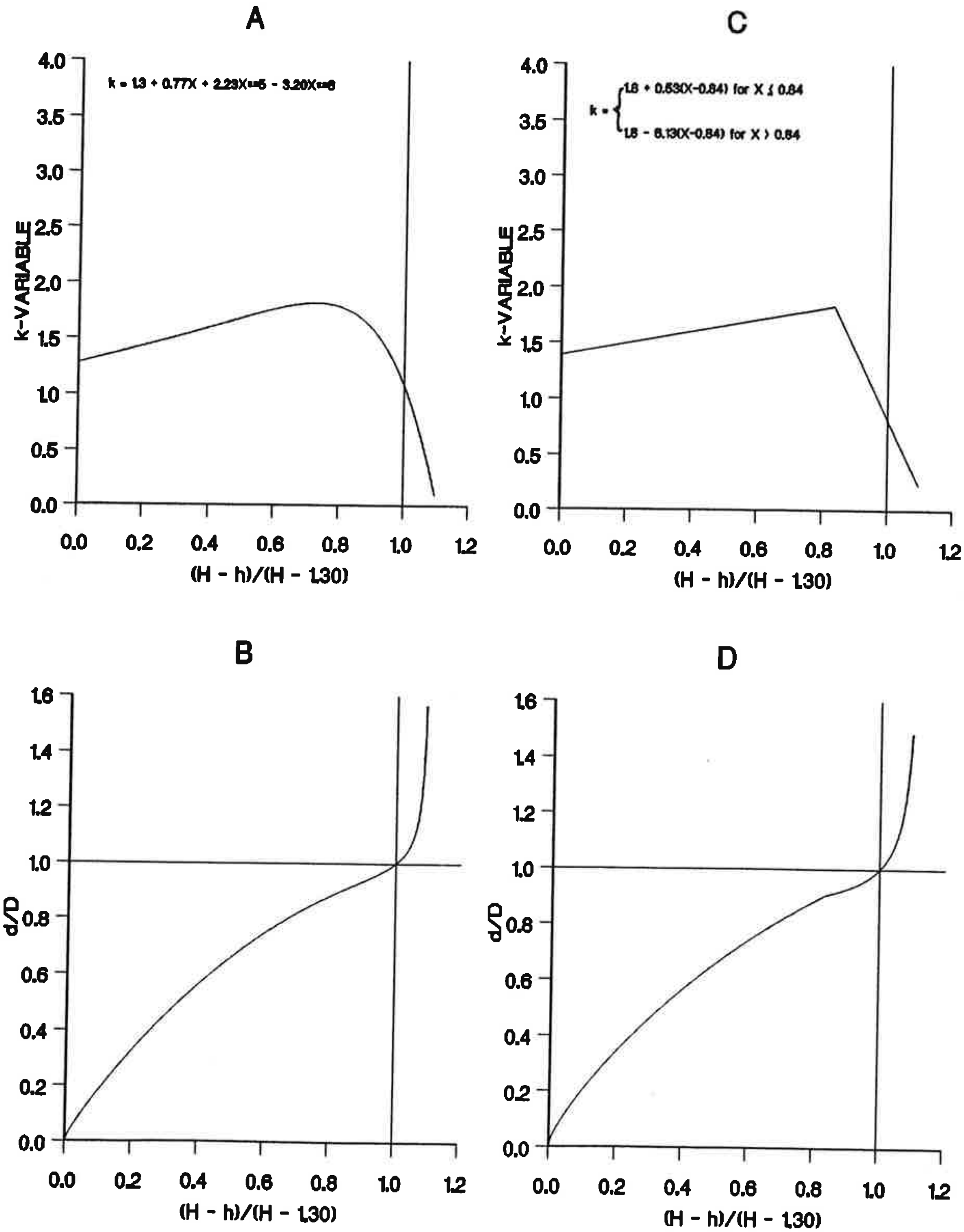

Figure 6. The relationship between k and relative height and the corresponding taper curves for Models 1 (A and B) and 2 (C and D).

Model 3. It is generally accepted that stem form varies with the amount of competition to which the tree is subjected from surrounding trees. Free-growing trees are conical in form with rapid taper, while stems of trees growing in forest stands tend to be conical within the crown and paraboloidal for the greater portion of the stem below the crown (see, for example, Figure 4). The length of the conical portion will thus be largely governed by the length of the crown, or the live crown ratio (C), so that it would appear logical to include C in any regression model for k. Although estimates of C were available for the red pine data, this is usually not the case in practice. However, DBH and total height (either measured directly or estimated from DBH) are generally available and the ratio D/H is a good indicator of the live crown ratio (for a given height, free-growing trees with large crowns will have a greater value for D/H than forest-grown trees with relatively small crowns). The relationship between the D/H ratio and the live crown ratio for the 272 red pine trees in the working data set is shown in Figure 7.

For Model 3, besides the first six powers of X that were tested in Model 1, the variables D/H and $(D/H)^2$ were added. Also included were the combined variables $X.D/H$, $X^2.D/H$, $X^3.D/H$, $X.(D/H)^2$, and $X^2.(D/H)^2$. The best three-variable regression was:

$$[9] \quad k = 2.48 - 1.540X^6 - 0.696(D/H) + 0.770X^2.(D/H) \qquad R^2 = 0.6688$$

Thus the inclusion of the D/H ratio in the regression has accounted for a significant increase in the value of R^2 (R^2 for Model 1 was only 0.5654). For the red pine data, similar results were obtained when the live crown ratio was used instead of D/H although the value of R^2 (0.6530) was slightly lower (see Model 15 in Appendix I). Model 3 is illustrated in Figure 8A together with the corresponding taper curves for D/H ratios of 1.1, 1.5, and 1.9 (Figure 8B).

Model 3 was reworked as a nonlinear regression of the form:

$$[10] \quad k = b_0 + b_1 X^{b_2} + b_3 (D/H) + b_4 X^2.(D/H)$$

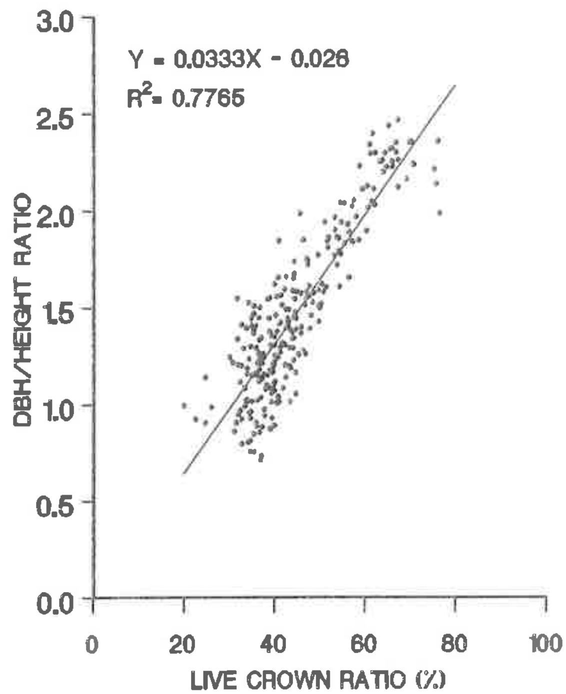

Figure 7. The relationship between the D/H ratio and the live crown ratio (expressed as a percentage) for the red pine working data set.

The values of the b-coefficients differed in value (but not in sign) from the corresponding coefficients in [9] (see Model 24 in Appendix I). The value of b_2 was 6.707, indicating that perhaps a higher power of X could have been included in Model 3.

Model 4. The variable X^6 is included in Model 3 largely to allow for the neiloidal butt-swell portion of the stem. There is a danger in including such large exponents, particularly if the model is used to extrapolate for data outside the range of the original data, as the estimated values may differ noticeably from those expected. This could, in the extreme case, result in negative values for diameters close to the stump.

In Model 4, X^6 has been replaced by $1/h$, a variable that will have values that are relatively high in the stump region but that decline rapidly with increase in height. As the values of k are smallest in the butt portion of the stem, the coefficient for $1/h$ should be negative. The three-variable regression was:

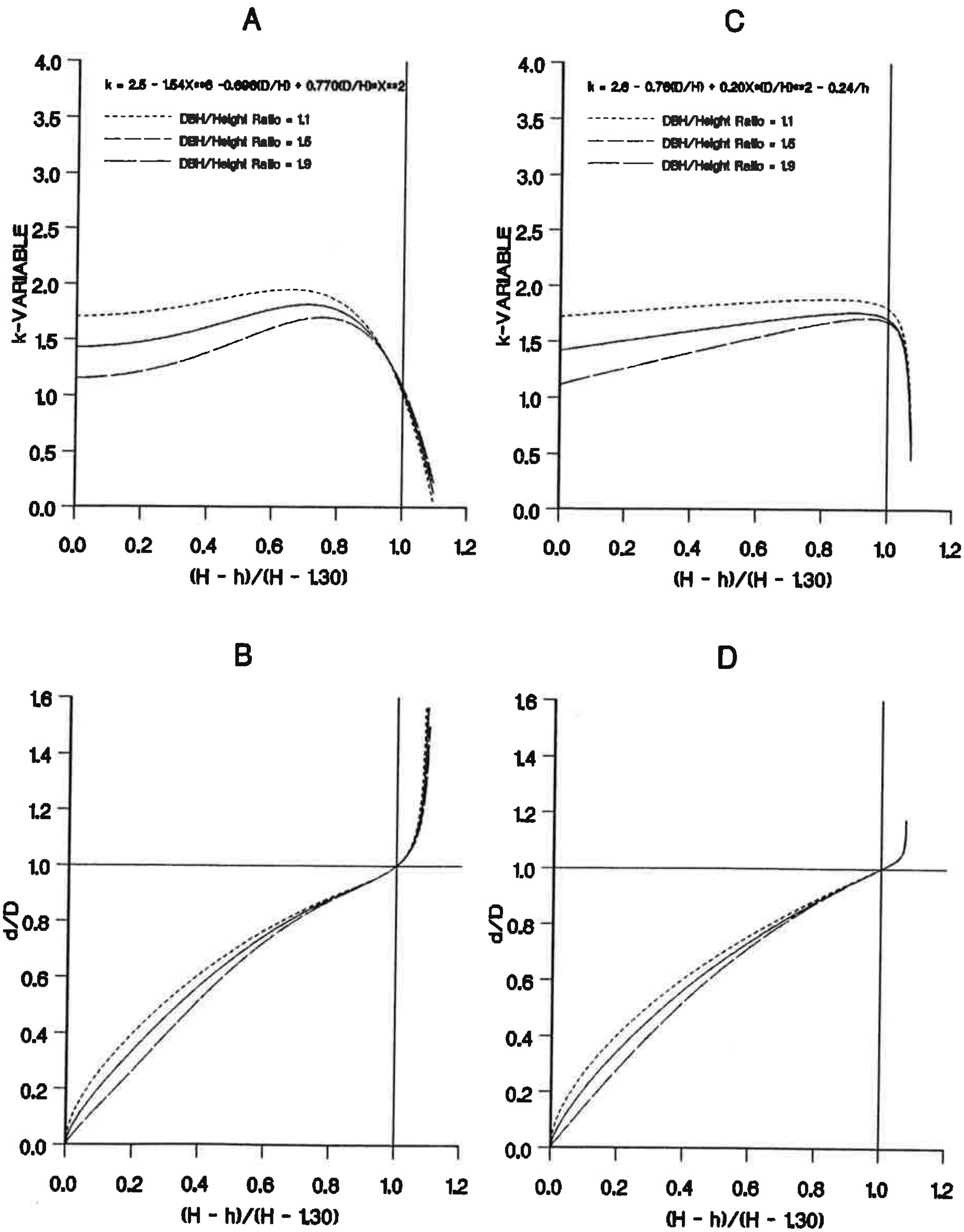

Figure 8. The relationship between k and relative height and the corresponding taper curves for Models 3 (A and B) and 4 (C and D).

[11] $k = 2.58 - 0.763(D/H) + 0.205X.(D/H)^2 - 0.244(1/h) \qquad R^2 = 0.6579$

This is similar in form to Model 3 but it now contains no terms with an exponent greater than 2. It is therefore less likely to cause the problems that could occur with Model 3.

Model 4 is illustrated in Figure 8 for the tree of mean height (16.43 m) and for D/H ratios of 1.1, 1.5, and 1.9. Model 4 was also reworked as a non-linear regression of the form:

[12] $k = b_0+b_1 (D/H)+b_2X.(D/H)^2+b_3 (1/h)^{b_4}$

Again, although the value of the coefficients differed in magnitude, the signs were the same as in [11] (see Model 25 in Appendix I). However, the value of b_4 was 0.429, indicating that the square root of 1/h might be a better variable than 1/h to include in [11].

Model 5. Although not widely applied to date, the Max-Burkhart segmented polynomial regression model appears to be one of the best of the existing taper models. The Alberta Forest Service (1987) recommended it for use in Alberta. Other models, usually more complex in structure, may give better results under certain conditions but, in spite of this the Max-Burkhart model was chosen for comparison with the new models presented in this report.

For the 272 red pine trees, the Max-Burkhart segmented regression was:

[13] $$(d/D)^2 = -5.030(Z - 1) + 2.477(Z^2 - 1) - 2.577(0.7598 - Z)^2 I_1 + 154.8(0.0665 - Z)^2 I_2$$

where:

$I_1 = 1$, if $0 < Z \leq 0.7598$
 0, otherwise
$I_2 = 1$, if $0 < Z \leq 0.0665$
 0, otherwise

The taper curve for this model is illustrated in Figure 9.

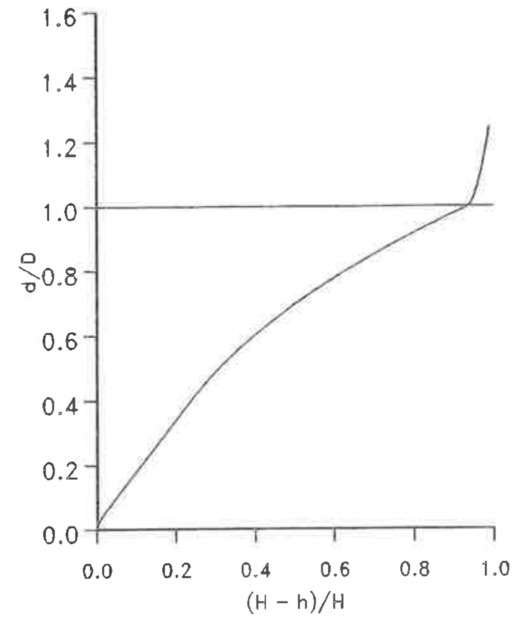

Figure 9. The taper curve for the Max-Burkhart segmented polynomial regression model.

Comparison of the Accuracy of the Five Models

Taper. Estimates of upper stem relative diameter (d/D) were obtained for each of the five models in turn. For each (H - h)/(H - 1.30) decile above breast height, and for observations below breast height, the average bias and the average error were calculated. Bias was the sum of the deviations (expected - observed), divided by the number of observations in the class. Average error was the sum of the absolute values of the deviations, divided by the number of observations.

The bias and average error for each of the models are shown for both the working and the control data sets in Figures 10 and 11, respectively. The overall bias and average error for each model are shown in Table 2 and, for all 25 models, in Appendix II.

For the working data set, all five models tend to overestimate stem diameter, particularly in the upper stem, the mid-decile and, for Models 1 and 3, the section below breast height. In the top decile, there is only one observation; possibly a better fit could have been obtained if more observations had been available for calculating the regressions. Bias in the second and third deciles from the top

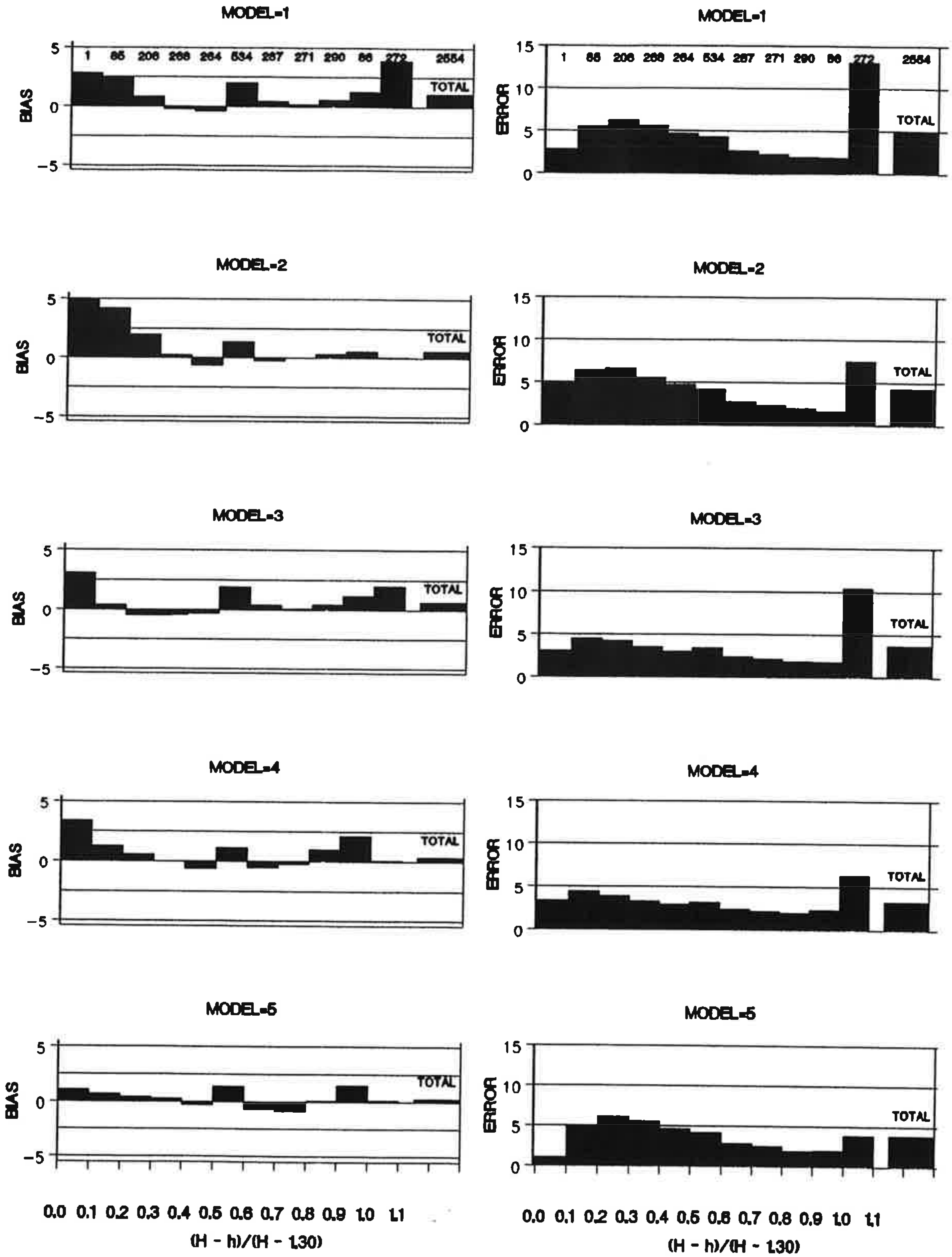

Figure 10. Bias (left) and average error (right) of estimates of relative diameter (d/D) for each relative height decile and for the whole tree for Models 1-5 (red pine working data set; units on the vertical scale are 100.d/D).

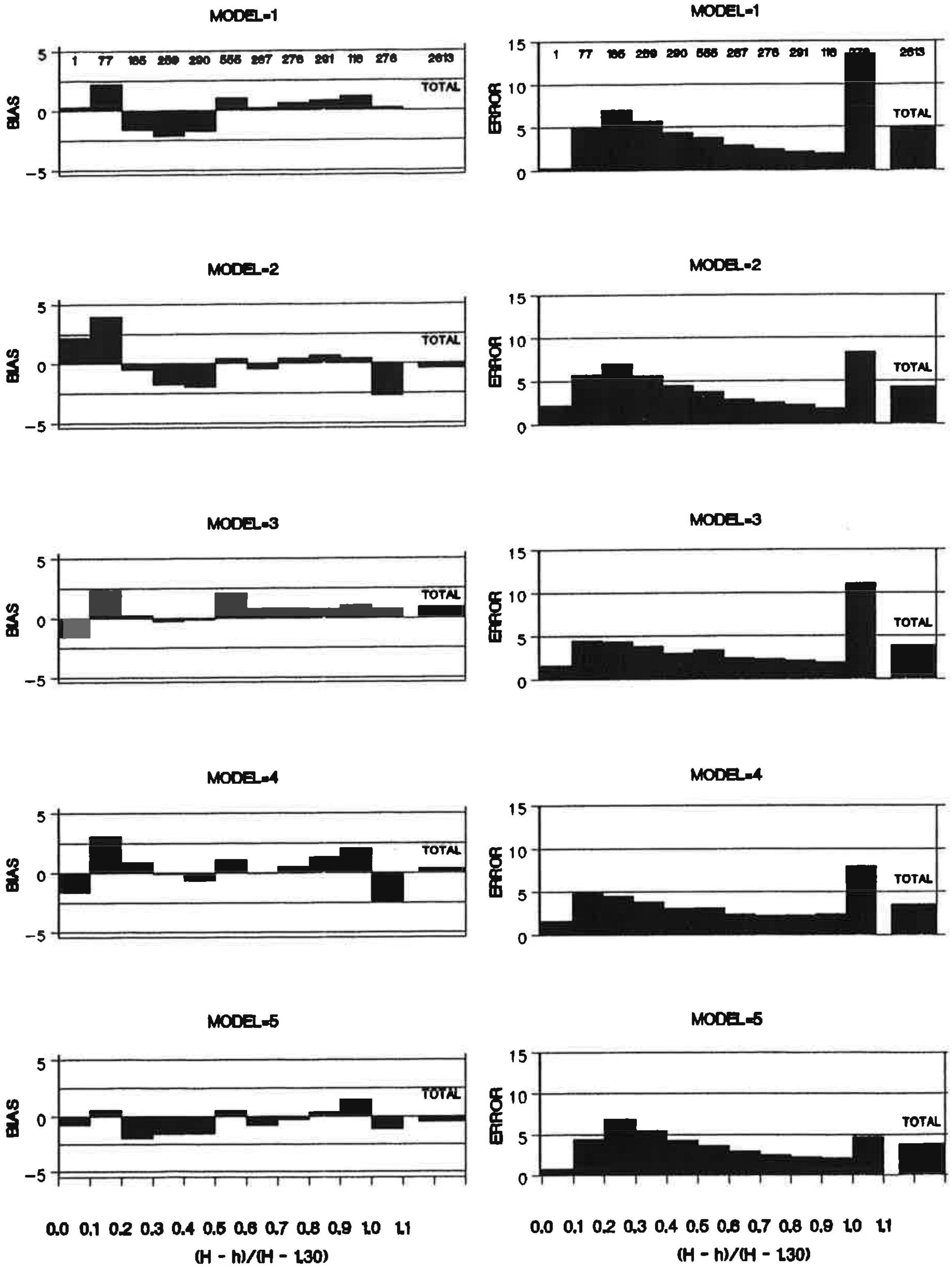

Figure 11. Bias (left) and average error (right) of estimates of relative diameter (d/D) for each relative height decile and for the whole tree for Models 1-5 (red pine control data set; units on the vertical scale are 100.d/D).

Table 2. Bias and average errors for estimating stem diameter ratios and total volume of red pine trees in the working and control data sets using Models 1–5

Model No.	Working data set				Control data set			
	d/D-ratio		Volume (%)		d/D-ratio		Volume (%)	
	Bias	Av. error	Bias	Av. error	Bias	Av. error	Bias	Av. error
1	0.0108	0.0486	3.82	7.31	0	0.0482	1.76	7.04
2	0.0057	0.0430	2.47	6.16	-0.0044	0.0429	0.51	6.01
3	0.0062	0.0373	-0.14	4.48	0.0086	0.0387	1.00	4.41
4	0.0035	0.0325	0.12	3.92	0.0034	0.0354	0.62	4.08
5	0.0023	0.0383	1.58	5.67	-0.0051	0.0383	0.20	5.38

NOTE: Volume bias and average error are expressed as percentages of the observed mean volumes (0.414 m^3 and 0.360 m^3 for the working and control data sets respectively).

of the tree could lead to errors in estimating merchantable height (and therefore merchantable volume) so that Models 1 and 2 should not be used for this purpose. Model 2, in spite of the theoretical disadvantage of a discontinuity in the taper curve (Figure 6D), gives remarkably unbiased estimates over much of the stem. Model 5, followed closely by Model 4, appear best with respect to producing unbiased estimates of stem diameters.

Average error tends to decrease down the stem until the section below breast height is reached where there is an increase in all cases, but one that is less noticeable for Model 5. Model 4 has the lowest overall average error.

The patterns for bias and average error are very similar for the control data set (Figure 11), except that estimates tend to be a little lower. There are sections of the stem with quite noticeable negative bias and, for Models 2 and 5, an overall negative bias. Average errors are similar to those of the working data set. Again, Models 4 and 5 appear to be best.

Total Volume. For each tree, the volume outside bark of each 5-ft (1.52 m) section and for the section between breast and stump height, was calculated using Smalian's formula. The volume of the stump was assumed to be that of a cylinder (of stump height diameter). The volume of the top section of the stem (above the last 5-ft section) was assumed to be that of a cone. These volumes were summed to give the total volume outside bark. Merchantable height was not recorded so that it was not possible to calculate merchantable volume.

To be comparable, volume estimates obtained using each taper model were calculated in the same way except that, instead of observed diameters, diameters estimated from the different models were used. For the three most promising models (Models 3 - 5), the bias in estimating total volume and the average error are given by 10 cm DBH class and 1 m height class for the working data set (Table 3). The overall bias and error are given for all five models, for both the working and control data sets, in Table 2.

There is a tendency for the three models to underestimate volume in the lowest (except in the lowest height class for Model 3) and upper DBH classes, and to overestimate in the 20 - 30 cm class. A noticeable exception is Model 5 that, while underestimating in the lowest DBH class, overestimates in the remaining classes with a trend of increasing bias with increasing DBH. This trend, coupled with a similar trend with increasing height, is surprising (and difficult to explain), as Model 5 was one of the best for predicting stem diameter. Model 4 has both the lowest overall bias (0.12 per cent) and average error (3.92 per cent).

Table 3. Bias and average error for estimating total volume for Models 3-5 by 10-cm DBH class and 1-m total height class (working data set)

Height class (m)		Model No.	DBH class (cm)				Total
			‹20	20-	30-	>40	
‹15	N		18	34	4	0	56
	Bias	3	9.42	4.93	-4.03		4.52
		4	-2.28	2.34	-2.25		0.85
		5	-7.92	1.70	3.17		0.12
	Av. Error	3	10.45	7.02	4.03		7.24
		4	4.11	4.67	3.35		4.38
		5	8.58	4.06	4.39		4.95
15-	N		18	29	15	0	62
	Bias	3	-0.91	0.10	-1.75		-0.79
		4	-4.23	-0.09	0.42		-0.55
		5	-8.56	-1.39	5.43		0.17
	Av. Error	3	3.80	2.84	5.41		4.01
		4	5.89	2.73	4.30		3.86
		5	9.28	3.05	5.87		5.16
16-	N		13	25	15	0	53
	Bias	3	-0.41	0.70	-2.27		-0.83
		4	-2.44	1.27	0.53		0.52
		5	-6.71	1.00	7.48		3.26
	Av. Error	3	3.06	4.08	4.28		4.07
		4	3.40	4.32	4.57		4.34
		5	6.71	4.55	8.15		6.49
17-	N		19	18	9	7	53
	Bias	3	-3.62	-1.32	-3.74	-4.22	-3.16
		4	-4.82	-0.71	-1.17	-0.69	-1.43
		5	-9.01	-1.11	5.10	9.15	2.38
	Av. Error	3	5.30	2.83	3.74	4.22	3.84
		4	5.74	2.81	2.82	2.22	3.07
		5	9.01	3.14	5.39	9.15	6.32
≥18	N		9	36	3	0	48
	Bias	3	-3.31	2.57	0.11		1.52
		4	-3.60	2.95	1.69		1.95
		5	-6.39	2.23	4.75		1.38
	Av. Error	3	3.58	4.46	1.15		3.98
		4	3.79	4.69	1.69		4.24
		5	6.39	4.81	4.75		5.01
Total	N		77	142	46	7	272
	Bias	3	0.22	1.70	-2.44	-4.22	-0.14
		4	-3.59	1.41	-0.02	-0.69	0.12
		5	-7.90	0.72	5.86	9.15	1.58
	Av. Error	3	5.35	4.40	4.26	4.22	4.48
		4	4.76	3.97	3.81	2.22	3.92
		5	8.20	4.02	6.37	9.15	5.67

NOTE: Volume bias and average error are expressed as percentages of the observed mean volume for the respective class combination.

Estimation of Merchantable Height

One disadvantage of the new taper function presented here is that while it is possible to estimate diameter for a given height directly from [6] the reverse is not true. This is because both k and d/D are functions of relative height, X. Height h_m for a given upper-stem merchantable diameter limit d_m can only be estimated by iteration. If it is necessary to do this for many thousands of trees, much expensive computer time may be required unless the first estimate of h_m gives an estimated diameter that is close to d_m. This first estimate may be obtained from a regression of h_m on D, or of h/H on d/D, if such regressions are available. In this study, the first estimate of h_m was obtained from a polynomial regression of (H - h)/(H - 1.30) on d/D and the H/D ratio for those observations above breast height in the working data set. The observations at and below breast height were discarded because it was assumed that, if d_m occurred below breast height, the tree would not be considered merchantable (i.e. a merchantable tree must contain at least one 1-m log). The regression was:

$$[14] \quad (H - h)/(H - 1.30) = 0.33 + 0.667(d/D)^2 - 0.414(H/D) + 0.384(dH/D^2) \qquad R^2 = 0.9540$$

The first estimate, h′, of h_m can be obtained by substituting d_m for d in [14] and transposing to give:

$$[15] \quad h' = H - (H - 1.30)[0.33 + 0.667(d_m/D) - 0.414(H/D) + 0.384(d_mH/D^2)]$$

Substitution of this value of h′ in the taper function for one of Models 1 - 4 will give a value, d′, that probably differs slightly from the specified value for d_m. A new estimate h″ is obtained from:

$$[16] \quad h'' = H - (H - h')(d_m/d')^k$$

This process is continued until a value of h is obtained that gives a value of d that differs from d_m by less than an allowable error (e.g. 0.005 cm) specified by the user.

This method was used with Model 3 to estimate the merchantable height to a 7.5 cm top diameter (outside bark -- approximately equal to the Ontario standard of 7.0 cm inside bark) for each tree in the working data set. Total and merchantable volumes were also calculated. For 272 trees, these calculations required 3.74 seconds of CPU time on a VAX/785 computer. The average number of iterations to obtain the final estimate of h_m was 3.4/tree. Similar results were obtained with the control data set.

As merchantable volume was not available for the basic data, it was not possible to compare estimated values with observed values.

Volume Estimation by Integration

It has already been noted that it is not possible to calculate directly the above-stump volume of the stem by integrating the taper function [6] between the top and stump level, or the merchantable volume by integrating between the limits of merchantability. The only way that this can be done is by iteration, whereby the stem is divided into short sections, d is estimated at the top and bottom of each section, and its volume calculated using Smalian's (or a similar) formula. The smaller the section, the more accurate will be the estimated volume. However, the greater the number of sections, the greater the amount of CPU time that will be required to complete the calculations.

To obtain an idea of the CPU time, the total volume of a tree with a DBH of 25 cm and a height above stump level of 16.5 m was calculated using both Models 3 and 4. Section volume V was calculated using the formula:

$$[17] \quad V = \frac{\pi D^2}{40\,000} \bullet \frac{X_L^{2/k+1} - X_U^{2/k+1}}{H^{2/k}(2/k+1)}$$

where X_U and X_L are the upper and lower diameters of the section and k is estimated from [9] (Model 3) or [11] (Model 4). To obtain more accurate estimates, CPU times were calculated for 100 trees over the range of section lengths given in Table 4. As would be expected, the CPU times decreased as the number of sections into which the stem was divided decreased (the time taken to calculate the volume of 1000 sections is approximately 0.3 seconds). Volume estimates are consistent to

Table 4. Estimated CPU times for calculating the volume above stump height for 100 trees by numerical integration using Models 3 and 4. Each tree has a DBH of 25.0 cm and a total height above stump height of 16.5 m.

No. of sections	Section length (m)	Model 3		Model 4	
		Volume (m^3)	CPU time (seconds)	Volume (m^3)	CPU time (seconds)
1650	0.010	0.356427	54.25	0.359332	44.58
1000	0.165	0.356428	32.84	0.359332	27.42
825	0.020	0.356427	26.95	0.359332	22.36
500	0.033	0.356427	16.45	0.359331	13.61
330	0.050	0.356425	10.88	0.359329	8.76
250	0.066	0.356424	8.17	0.359326	6.79
165	0.100	0.356418	5.36	0.359317	4.39

the fifth decimal place until the section length exceeds 0.05 m. Model 4 gives slightly greater volumes than Model 3. A section length of 0.05 m should provide accurate enough estimates of volume for most practical applications. This could probably be increased for larger trees (perhaps maintaining the number of sections at about 330/tree).

DISCUSSION AND CONCLUSIONS

The new taper model that has been presented here takes into account the fact that the geometric form of the tree stem varies continuously along its entire length. In this manner, form can be expressed by the variable k in the expression:

$$(d/D)^k = (H - h)/(H - 1.30)$$

The hypothesis presented earlier was that k could have values other than those associated with the conventional geometric solids --- cone (k = 1), paraboloid (k = 2), and neiloid (k = 2/3) --- that are described in forest mensuration textbooks and that have often been assumed in previous taper studies.

The main objective of the present study, to determine a satisfactory relationship between k and relative height along the stem, has been achieved. One problem encountered was that, although the value of k increases fairly smoothly with distance from the top of the stem, there is a sudden decrease at a point above breast height where the influence of butt-swell is manifested. A number of models were tested, of which four have been described in detail and the remainder given in Appendix I.

To account for the rapid decline in the value of k in the lower part of the stem, polynomial equations with the independent variable raised to the power of six were required (see [7] and [9]). Although powers as high (or higher) than this have been used in previous taper studies, their main disadvantage is that, at the extremes of the range (in the present case, just below stump height ie. 0.1524 m), estimates of the dependent variable can be subject to large errors and extrapolation beyond the range (dangerous in the best of circumstances) virtually impossible. To avoid the use of such high powers, the variable 1/h was tested and found to be a significant component of one of the best models (see [11]). However, there is still a danger with this variable as its value approaches infinity near ground level. Extremely large values (negative or positive) of both k and d/D can result.

This problem of obtaining a good taper curve at the base of the stem has plagued mensurationists for generations. Until recently, butt swell was often treated as an anomaly, the paraboloidal form of the stem was assumed to extend to ground level, and the excess stem wood due to butt swell was ignored. In the past 20 years, much of this previously wasted wood is now utilized for pulp, composite wood products, or energy. It has thus become more important to obtain accurate taper functions for this part of the stem.

In recent years, segmented polynomial equations have become popular while other researchers have developed very complex models. Many of these models contain five or more coefficients and contain variables raised to a power as high as 40. One of the best models is the Max-Burkhart segmented polynomial regression (see [13]). Although containing five parameters, it has no powers greater than two and, in this study, gave a satisfactory fit along the stem from the top to stump level.

The best of the models that have been presented here appears to be Model 4 (equation [11]). Written in a more conventional form as a taper equation, this becomes:

$$d = D.X^{1/[2.58 - 0.763(D/H) + 0.205X.(D/H)^2 - 0.244(1/h)]}$$

where X = (H - h)/(H - 1.30). Another that merits further study is Model 3 (equation [9]):

$$d = D.X^{1/[2.48 - 1.540X^6 - 0.696(D/H) + 0.770X^2.(D/H)]}$$

The advantages of both these models are:

(1) They give consistently accurate estimates of diameter along the stem and of total volume for both the working and control data sets of the plantation-grown red pine used in this study. Model 4 is the better in this respect and both compare favourably with the results obtained from the Max-Burkhart model.
(2) They take into account the continuous variation in form along the stem of the tree and thus require only a single equation for the taper curve.
(3) They require only four parameters in the regression equation.
(4) They take into account variations in stem form that are related to crown size, as represented by the D/H ratio.
(5) By using standard linear regression (least-squares) techniques computing time is reduced and some problems in obtaining a solution, that may occur when nonlinear models are used, are avoided.

The disadvantages of Models 3 and 4 are:

(1) Both are very sensitive to errors in estimating stem diameter near the base of the tree. With Model 4, for example, diameter reaches a maximum a few cm below stump height and then falls rapidly to a negative value before rising again. The accuracy of the regression equation in this region could probably be improved if one or, preferably, two additional measurements had been taken between stump and breast heights (say, at 0.80 and 0.30 m). This problem may well occur with other taper functions, particularly where high-order powers are used.
(2) It is not possible to transpose either equation so that the height for a given diameter cannot be estimated directly. This means that, for example, the height at which a given upper merchantable diameter limit occurs must be estimated by iteration. However, a method of doing this efficiently has been described.
(3) It is not possible to integrate the equations for either model so that volume, either merchantable or total, can be estimated directly. Potentially time consuming numerical methods have to be employed, although preliminary testing with the red pine data indicates that this may not be so difficult a task as might be envisioned.

This study has shown that taper functions that take into account the continuous variation in form along the stem, such as Models 3 and 4, provide a simpler and more accurate solution to taper curve definition than do most existing taper functions and certainly merit further testing.

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APPENDIX I

Complete list of taper models tested in the study

Complete List of Taper Models

Definitions

Y	=	relative diameter, d/D
X	=	relative height, (H-h)/(H-1.30)
Z	=	h/H
W	=	exp (X)-1
D	=	diameter at breast height -- DBH (1.30 m)
H	=	total height
h	=	height above ground level
d	=	diameter at height, h
k	=	ln(X)/ln(Y)
C	=	live crown ratio (%)

Model 5 (Max-Burkhart)

$$Y^2 = -5.030\,(Z-1) + 2.477\,(Z^2 - 1) - 2.577\,(0.7598 - Z)^2 I_1 + 154.8\,(0.0665 - Z)^2\, I_2$$

where $I_1 = 1$, if $0 \leq Z \leq 0.7598$; 0, otherwise

$I_2 = 1$, if $0 \leq Z \leq 0.0665$; 0, otherwise

For Models 1–4, 6–10, 13–21, and 23–25, the general form of the model is:

$$Y = X^{1/k}$$

where k is a function of X and other independent variables.

Model

Model		
1	$k = 1.28 + 0.772X + 2.234X^5 - 3.198X^6$	$R^2 = 0.5654$
2	$k = \begin{cases} 1.83 + 0.526\,(X - 0.836) & \text{for } X \leq 0.836 \\ 1.83 - 6.133\,(X - 0.836) & \text{for } X > 0.836 \end{cases}$	$I^2 = 0.5611$*
3	$k = 2.48 - 1.540X^6 - 0.696(D/H) + 0.770X^2.(D/H)$	$R^2 = 0.6688$
4	$k = 2.58 - 0.763(D/H) + 0.205X.(D/H)^2 - 0.244(1/h)$	$R^2 = 0.6579$
6	$k = 1.38 + 1.291X^2 - 1.617X^6$	$R^2 = 0.5644$
7	$k = 1.39 + 1.200W^2 - 0.761W^3$	$R^2 = 0.5642$
8	$k = 1.50 + 1.955W^4 - 2.196W^5 + 0.591W^6$	$R^2 = 0.5677$
9	$k = 1.39 - 0.0157W + 1.216W^2 - 0.766W^3$	$R^2 = 0.5642$
10	$k = \begin{cases} 1.92 + 0.495\,(W-1.152) & \text{for } Z \leq 1.152 \\ 1.92 - 1.864\,(W - 1.152) & \text{for } Z > 1.152 \end{cases}$	$I^2 = 0.5493$*
13	$k = \begin{cases} 1.93 + 0.613\,(X^2 - 0.710) & \text{for } X^2 \leq 0.710 \\ 1.93 - 3.517\,(X^2 - 0.710) & \text{for } X^2 > 0.710 \end{cases}$	$I^2 = 0.5650$*
14	$k = 2.50 - 1.562X^6 - 0.0237C + 0.0260X^2.C$	$R^2 = 0.6530$
15	$k = 2.68 - 1.068X^8 - 0.0256C + 0.000301X^2.C^2$	$R^2 = 0.6530$
16	$k = 2.62 - 1.057X^8 - 0.734(D/H) + 0.266X^2.(D/H)^2$	$R^2 = 0.6688$

17[δ]	k	=	$1.0 + 1.908X - 1.728X^5 - 0.0000814C^2$	$I^2 = 0.5892$*
18[δ]	k	=	$1.0 - 0.535X^{12} + 0.0385C - 0.000487C^2$	$I^2 = 0.5489$*
19[δ]	k	=	$1.0 + 1.920X - 1.733X^5 - 0.0771(D/H)^2$	$I^2 = 0.5985$*
20[δ]	k	=	$1.0 - 0.530X^{12} - 0.0809(D/H)^2 + 0.0530H$	$I^2 = 0.5505$*
21	k	=	$1.51 + 1.819X^2 - 1.719X^6 - 0.360X.(D/H)$	$R^2 = 0.5982$
23	k	=	$2.27 - 0.377(D/H) - 0.203\ (1/h)$	$R^2 = 0.6084$
24[‡]	k	=	$2.42 - 1.382X^{6.707} - 0.657(D/H) + 0.681X^2.(D/H)$	$I^2 = 0.67021$
25[‡]	k	=	$3.05 - 0.898(D/H) + 0.276X.(D/H)^2 - 0.922(1/h)^{0.429}$	$I^2 = 0.66411$

Notes:

* I^2 is the correlation index: $1 - \Sigma(k-\hat{k})^2/\Sigma(k-\bar{k})^2$

δ Models 17–19 are constrained to pass through the (0,1) point

‡ Models 24 and 25 are nonlinear regressions with k as the dependent variable.

Models 11 and 12 are nonlinear regression models with Y as the dependent variable.

Model

11 $Y = X^{1/(1.44 + 0.666X^2 - 1.145X^6)}$ $I^2 = 0.9430$

12 $Y = X^{1/(1.77 - 5.831W^2 + 5.879W^3)}$ $I^2 = 0.6795$

where $I^2 = 1-\Sigma(Y-Y)^2/\Sigma(Y-Y)^2$

Model 22 is of the form:

$$Y = X^m$$

where $m = \ln(Y)/\ln(X)$ is a function of X and other independent variables

$$m = 0.68 - 8.037X^4 + 5.983X^6 + 0.186H \qquad R^2 = 0.8951$$

Because the dependent variable m is different, the value of R^2 cannot be compared with the R^2 values for the group of models that have k as a dependent variable.

APPENDIX II

Bias and Average Error by Height Decile for 25 Taper Models (Working Data Set)

1. Bias and average error by height decile for 25 taper models (working data set).

Eqn. No.	0.0-	0.1-	0.2-	0.3-	0.4-	0.5-	0.6-	0.7-	0.8-	0.9-	>1.0	Total
	Height Decile, (H - H)/(H - 1.30)											
1	1	85	206	268	264	534	287	271	280	86	272	2554
Bias	0.0280	0.0236	0.0079	-0.0022	-0.0039	0.0199	0.0044	0.0018	0.0056	0.0127	0.0388	0.010
Av Er	0.0280	0.0549	0.0623	0.0559	0.0468	0.0429	0.0270	0.0227	0.0190	0.0187	0.1306	0.048
2	1	85	206	268	264	534	287	271	280	86	272	2554
Bias	0.0482	0.0417	0.0198	0.0023	-0.0061	0.0134	-0.0024	0.0000	0.0032	0.0054	0.0005	0.005
Av Er	0.0482	0.0637	0.0657	0.0557	0.0472	0.0415	0.0271	0.0227	0.0195	0.0161	0.0751	0.043
3	1	85	206	268	264	534	287	271	280	86	272	2554
Bias	0.0305	0.0037	-0.0048	-0.0047	-0.0034	0.0189	0.0035	0.0007	0.0041	0.0112	0.0198	0.006
Av Er	0.0305	0.0448	0.0419	0.0350	0.0295	0.0339	0.0235	0.0213	0.0184	0.0179	0.1043	0.037
4	1	85	206	268	264	534	287	271	280	86	272	2554
Bias	0.0336	0.0125	0.0055	-0.0001	-0.0063	0.0109	-0.0051	-0.0023	0.0099	0.0209	0.0007	0.003
Av Er	0.0336	0.0446	0.0390	0.0333	0.0296	0.0319	0.0241	0.0213	0.0195	0.0235	0.0635	0.032
5	1	85	206	268	264	534	287	271	280	86	272	2554
Bias	0.0104	0.0069	0.0040	0.0024	-0.0032	0.0134	-0.0069	-0.0086	0.0005	0.0147	0.0009	0.002
Av Er	0.0104	0.0492	0.0612	0.0552	0.0463	0.0412	0.0278	0.0246	0.0192	0.0198	0.0380	0.038
6	1	85	206	268	264	534	287	271	280	86	272	2554
Bias	0.0387	0.0282	0.0065	-0.0047	-0.0048	0.0207	0.0056	0.0023	0.0051	0.0119	0.0241	0.009
Av Er	0.0387	0.0570	0.0622	0.0561	0.0469	0.0431	0.0271	0.0227	0.0190	0.0183	0.1164	0.047
7	1	85	206	268	264	534	287	271	280	86	272	2554
Bias	0.0387	0.0282	0.0065	-0.0047	-0.0047	0.0207	0.0055	0.0022	0.0052	0.0120	0.0267	0.009
Av Er	0.0387	0.0570	0.0622	0.0561	0.0469	0.0431	0.0271	0.0227	0.0190	0.0184	0.1195	0.047
8	1	85	206	268	264	534	287	271	280	86	272	2554
Bias	0.0622	0.0490	0.0172	-0.0053	-0.0113	0.0154	0.0060	0.0058	0.0058	0.0103	-0.0112	0.005
Av Er	0.0622	0.0677	0.0651	0.0562	0.0483	0.0418	0.0271	0.0230	0.0193	0.0176	0.0573	0.041
9	1	85	206	268	264	534	287	271	280	86	272	2554
Bias	0.0393	0.0287	0.0067	-0.0047	-0.0048	0.0206	0.0055	0.0022	0.0052	0.0121	0.0273	0.009
Av Er	0.0393	0.0572	0.0622	0.0561	0.0469	0.0431	0.0271	0.0227	0.0190	0.0184	0.1199	0.047
10	1	85	206	268	264	534	287	271	280	86	272	2554
Bias	0.0379	0.0310	0.0111	-0.0025	-0.0062	0.0174	0.0047	0.0055	-0.0026	0.0012	-0.0668	-0.001
Av Er	0.0379	0.0584	0.0631	0.0559	0.0472	0.0422	0.0270	0.0229	0.0198	0.0156	0.0749	0.042
11	1	85	206	268	264	534	287	271	280	86	272	2554
Bias	0.0503	0.0384	0.0107	-0.0090	-0.0171	0.0032	-0.0135	-0.0147	-0.0064	0.0034	-0.0473	-0.008
Av Er	0.0503	0.0620	0.0632	0.0565	0.0499	0.0409	0.0301	0.0262	0.0200	0.0158	0.0800	0.044
12	1	85	206	268	264	534	287	271	280	86	272	2554
Bias	0.1071	0.0752	-0.0075	-0.1034	-0.1788	-0.1535	-0.0473	0.0464	0.0623	0.0612	-0.2456	-0.077
Av Er	0.1071	0.0866	0.0651	0.1099	0.1791	0.1540	0.0586	0.0485	0.0623	0.0612	0.2456	0.117
13	1	85	206	268	264	534	287	271	280	86	272	2554
Bias	0.0605	0.0494	0.0213	0.0005	-0.0083	0.0129	-0.0004	0.0039	0.0088	0.0114	0.0096	0.008
Av Er	0.0605	0.0678	0.0663	0.0558	0.0477	0.0414	0.0269	0.0229	0.0201	0.0181	0.0864	0.044
14	1	85	206	268	264	534	287	271	280	86	272	2554
Bias	0.0346	0.0036	-0.0013	-0.0054	-0.0032	0.0193	0.0038	0.0010	0.0044	0.0114	0.0316	0.007
Av Er	0.0346	0.0461	0.0409	0.0351	0.0308	0.0353	0.0242	0.0217	0.0187	0.0181	0.1215	0.0397
15	1	85	206	268	264	534	287	271	280	86	272	2554
Bias	0.0536	0.0208	0.0122	0.0011	-0.0035	0.0151	-0.0007	-0.0009	0.0052	0.0127	0.1189	0.0181
Av Er	0.0536	0.0521	0.0438	0.0347	0.0306	0.0338	0.0245	0.0218	0.0186	0.0186	0.2102	0.049
16	1	85	206	268	264	534	287	271	280	86	272	2554
Bias	0.0487	0.0210	0.0083	0.0014	-0.0038	0.0147	-0.0008	-0.0010	0.0052	0.0130	0.1472	0.0207
Av Er	0.0487	0.0495	0.0416	0.0345	0.0293	0.0325	0.0235	0.0212	0.0184	0.0183	0.2376	0.0513
17	1	85	206	268	264	534	287	271	280	86	272	2554
Bias	-0.0531	-0.0609	-0.0416	-0.0199	-0.0023	0.0271	0.0100	0.0029	0.0031	0.0104	31.4339	3.3477
Av Er	0.0531	0.0646	0.0539	0.0452	0.0375	0.0410	0.0253	0.0219	0.0189	0.0180	31.5726	3.394

1. (continued).

Eqn. No.	0.0-	0.1-	0.2-	0.3-	0.4-	0.5-	0.6-	0.7-	0.8-	0.9-	>1.0	Total
	Height Decile, (H - h)/(H - 1.30)											
18	1	85	206	268	264	534	287	271	280	86	272	2554
Bias	0.1089	0.0737	0.0491	0.0223	0.0034	0.0124	-0.0103	-0.0111	-0.0003	0.0119	5241.0449	558.1791
Av Er	0.1089	0.0830	0.0697	0.0481	0.0382	0.0363	0.0281	0.0250	0.0190	0.0190	5241.2300	558.2215
19	1	85	206	268	264	534	287	271	280	86	272	2554
Bias	-0.0560	-0.0627	-0.0448	-0.0207	-0.0028	0.0268	0.0099	0.0030	0.0032	0.0108	0.1606	0.0167
Av Er	0.0560	0.0651	0.0569	0.0444	0.0358	0.0397	0.0247	0.0217	0.0186	0.0180	0.2751	0.0605
20	1	85	206	268	264	534	287	271	280	86	272	2554
Bias	0.0586	0.0588	0.0431	0.0217	0.0039	0.0115	-0.0109	-0.0113	0.0001	0.0150	164.2687	17.5032
Av Er	0.0586	0.0669	0.0607	0.0454	0.0362	0.0352	0.0282	0.0256	0.0192	0.0203	164.5617	17.5575
21	1	85	206	268	264	534	287	271	280	86	272	2554
Bias	0.0551	0.0385	0.0100	-0.0049	-0.0065	0.0181	0.0039	0.0018	0.0052	0.0137	2.2982	0.2511
Av Er	0.0551	0.0594	0.0568	0.0483	0.0395	0.0370	0.0238	0.0214	0.0188	0.0193	2.5420	0.3018
22	1	85	206	268	264	534	287	271	280	86	272	2554
Bias	0.0557	0.0388	-0.0009	-0.0301	-0.0354	0.0049	0.0138	0.0175	-0.0033	-0.0286	-0.0022	-0.0027
Av Er	0.0557	0.0623	0.0624	0.0628	0.0600	0.0468	0.0379	0.0336	0.0231	0.0297	0.0398	0.0453
23	1	85	206	268	264	534	287	271	280	86	272	2554
Bias	0.0914	0.0735	0.0473	0.0225	0.0033	0.0115	-0.0112	-0.0111	0.0026	0.0162	2.7322	0.3008
Av Er	0.0914	0.0769	0.0596	0.0411	0.0317	0.0323	0.0264	0.0239	0.0187	0.0210	2.8581	0.3344
24	1	85	206	268	264	534	287	271	280	86	272	2554
Bias	0.0334	0.0073	-0.0031	-0.0052	-0.0051	0.0172	0.0028	0.0014	0.0056	0.0130	0.0357	0.0078
Av Er	0.0334	0.0441	0.0408	0.0346	0.0296	0.0334	0.0235	0.0213	0.0186	0.0187	0.1187	0.0387
25	1	85	206	268	264	534	287	271	280	86	272	2554
Bias	0.0278	0.0083	0.0067	0.0038	-0.0023	0.0135	-0.0042	-0.0033	0.0077	0.0185	-0.0001	0.0044
Av Er	0.0278	0.0475	0.0399	0.0333	0.0290	0.0326	0.0240	0.0217	0.0188	0.0216	0.0572	0.0320

2. Bias and average error by height decile for 25 taper models (control data set).

Eqn. No.	0.0-	0.1-	0.2-	0.3-	0.4-	0.5-	0.6-	0.7-	0.8-	0.9-	>1.0	Total
	Height Decile, (H - h)/(H - 1.30)											
1	1	77	185	259	290	555	287	276	291	116	276	2613
Bias	0.0029	0.0219	-0.0168	-0.0220	-0.0178	0.0102	0.0021	0.0062	0.0080	0.0117	0.0022	0.000[illegible]
Av Er	0.0029	0.0483	0.0699	0.0570	0.0440	0.0379	0.0286	0.0246	0.0221	0.0194	0.1346	0.048[illegible]
2	1	77	185	259	290	555	287	276	291	116	276	2613
Bias	0.0217	0.0397	-0.0053	-0.0175	-0.0199	0.0037	-0.0046	0.0042	0.0064	0.0042	-0.0273	-0.004[illegible]
Av Er	0.0217	0.0575	0.0700	0.0558	0.0449	0.0375	0.0288	0.0244	0.0221	0.0177	0.0829	0.042[illegible]
3	1	77	185	259	290	555	287	276	291	116	276	2613
Bias	-0.0159	0.0226	0.0023	-0.0026	-0.0018	0.0206	0.0080	0.0081	0.0075	0.0106	0.0077	0.0086
Av Er	0.0159	0.0446	0.0436	0.0381	0.0295	0.0337	0.0247	0.0231	0.0216	0.0187	0.1109	0.0387
4	1	77	185	259	290	555	287	276	291	116	276	2613
Bias	-0.0162	0.0305	0.0090	-0.0010	-0.0066	0.0114	-0.0007	0.0050	0.0132	0.0207	-0.0246	0.0034
Av Er	0.0162	0.0478	0.0449	0.0381	0.0305	0.0308	0.0243	0.0225	0.0227	0.0238	0.0797	0.0354
5	1	77	185	259	290	555	287	276	291	116	276	2613
Bias	-0.0080	0.0053	-0.0198	-0.0164	-0.0160	0.0047	-0.0082	-0.0034	0.0035	0.0145	-0.0119	-0.0051
Av Er	0.0080	0.0440	0.0688	0.0546	0.0427	0.0367	0.0291	0.0252	0.0222	0.0209	0.0465	0.0383
6	1	77	185	259	290	555	287	276	291	116	276	2613
Bias	0.0158	0.0260	-0.0183	-0.0245	-0.0188	0.0110	0.0034	0.0067	0.0075	0.0109	-0.0100	-0.0014
Av Er	0.0158	0.0503	0.0699	0.0578	0.0443	0.0379	0.0287	0.0247	0.0221	0.0191	0.1225	0.0472
7	1	77	185	259	290	555	287	276	291	116	276	2613
Bias	0.0159	0.0260	-0.0184	-0.0245	-0.0187	0.0110	0.0032	0.0066	0.0075	0.0110	-0.0079	-0.0012
Av Er	0.0159	0.0502	0.0699	0.0577	0.0443	0.0379	0.0287	0.0247	0.0221	0.0191	0.1252	0.0474
8	1	77	185	259	290	555	287	276	291	116	276	2613
Bias	0.0375	0.0463	-0.0082	-0.0250	-0.0252	0.0057	0.0037	0.0102	0.0085	0.0091	-0.0362	-0.0043
Av Er	0.0375	0.0618	0.0700	0.0579	0.0470	0.0375	0.0288	0.0254	0.0224	0.0186	0.0679	0.0420
9	1	77	185	259	290	555	287	276	291	116	276	2613
Bias	0.0165	0.0264	-0.0182	-0.0245	-0.0188	0.0109	0.0032	0.0066	0.0076	0.0110	-0.0074	-0.0011
Av Er	0.0165	0.0504	0.0699	0.0577	0.0443	0.0379	0.0287	0.0247	0.0221	0.0192	0.1256	0.0475
10	1	77	185	259	290	555	287	276	291	116	276	2613
Bias	0.0128	0.0290	-0.0138	-0.0222	-0.0201	0.0077	0.0025	0.0101	0.0002	0.0002	-0.0895	-0.0110
Av Er	0.0128	0.0516	0.0699	0.0571	0.0449	0.0376	0.0287	0.0254	0.0215	0.0177	0.0946	0.0441
11	1	77	185	259	290	555	287	276	291	116	276	2613
Bias	0.0265	0.0360	-0.0145	-0.0288	-0.0309	-0.0065	-0.0158	-0.0104	-0.0044	0.0025	-0.0730	-0.0185
Av Er	0.0265	0.0554	0.0699	0.0592	0.0496	0.0383	0.0317	0.0260	0.0215	0.0177	0.0947	0.0455
12	1	77	185	259	290	555	287	276	291	116	276	2613
Bias	0.0835	0.0705	-0.0359	-0.1229	-0.1910	-0.1625	-0.0489	0.0496	0.0655	0.0596	-0.2601	-0.0860
Av Er	0.0835	0.0805	0.0744	0.1258	0.1910	0.1626	0.0584	0.0518	0.0656	0.0596	0.2601	0.1252
13	1	77	185	259	290	555	287	276	291	116	276	2613
Bias	0.0354	0.0470	-0.0040	-0.0193	-0.0221	0.0031	-0.0027	0.0080	0.0118	0.0102	-0.0201	-0.0024
Av Er	0.0354	0.0622	0.0702	0.0563	0.0458	0.0375	0.0287	0.0250	0.0234	0.0188	0.0933	0.0445
14	1	77	185	259	290	555	287	276	291	116	276	2613
Bias	-0.0148	0.0285	0.0007	-0.0045	-0.0042	0.0193	0.0074	0.0080	0.0076	0.0106	0.0013	0.0072
Av Er	0.0148	0.0458	0.0514	0.0416	0.0321	0.0348	0.0261	0.0242	0.0220	0.0190	0.1153	0.0410
15	1	77	185	259	290	555	287	276	291	116	276	2613
Bias	-0.0003	0.0471	0.0149	0.0028	-0.0037	0.0155	0.0030	0.0057	0.0080	0.0118	0.1017	0.0187
Av Er	0.0003	0.0582	0.0549	0.0424	0.0318	0.0333	0.0254	0.0237	0.0220	0.0193	0.2115	0.0513
16	1	77	185	259	290	555	287	276	291	116	276	2613
Bias	-0.0013	0.0402	0.0154	0.0039	-0.0016	0.0169	0.0040	0.0062	0.0082	0.0124	0.1035	0.0196
Av Er	0.0013	0.0539	0.0461	0.0379	0.0297	0.0321	0.0241	0.0226	0.0215	0.0191	0.2177	0.0500
17	1	77	185	259	290	555	287	276	291	116	276	2613
Bias	-0.0750	-0.0497	-0.0536	-0.0313	-0.0109	0.0220	0.0114	0.0099	0.0074	0.0102	12.4934	1.3183
Av Er	0.0750	0.0543	0.0686	0.0531	0.0374	0.0380	0.0282	0.0247	0.0220	0.0188	12.6663	1.3708

2. (continued).

Eqn. No.	0.0-	0.1-	0.2-	0.3-	0.4-	0.5-	0.6-	0.7-	0.8-	0.9-	>1.0	Total
	Height Decile, (H - h)/(H - 1.30)											
18	1	77	185	259	290	555	287	276	291	116	276	2613
Bias	0.0666	0.0908	0.0368	0.0129	-0.0014	0.0108	-0.0058	-0.0021	0.0048	0.0120	32.1094	3.4005
Av Er	0.0666	0.0976	0.0762	0.0514	0.0379	0.0359	0.0273	0.0240	0.0215	0.0195	32.3097	3.4468
19	1	77	185	259	290	555	287	276	291	116	276	2613
Bias	-0.0769	-0.0541	-0.0544	-0.0313	-0.0103	0.0226	0.0119	0.0103	0.0077	0.0111	0.0004	-0.0011
Av Er	0.0769	0.0559	0.0671	0.0512	0.0351	0.0369	0.0271	0.0239	0.0215	0.0185	0.1682	0.0497
20	1	77	185	259	290	555	287	276	291	116	276	2613
Bias	0.0361	0.0631	0.0302	0.0153	0.0015	0.0117	-0.0056	-0.0018	0.0054	0.0159	>99.9999	>99.9999
Av Er	0.0361	0.0706	0.0614	0.0450	0.0333	0.0333	0.0277	0.0251	0.0219	0.0210	>99.9999	>99.9999
21	1	77	185	259	290	555	287	276	291	116	276	2613
Bias	0.0327	0.0373	-0.0109	-0.0202	-0.0160	0.0130	0.0059	0.0093	0.0099	0.0144	0.5150	0.0571
Av Er	0.0327	0.0544	0.0640	0.0512	0.0381	0.0341	0.0255	0.0233	0.0218	0.0197	0.7563	0.1111
22	1	77	185	259	290	555	287	276	291	116	276	2613
Bias	0.0323	0.0359	-0.0268	-0.0518	-0.0534	-0.0109	0.0043	0.0139	-0.0063	-0.0339	-0.0189	-0.0165
Av Er	0.0323	0.0556	0.0716	0.0719	0.0660	0.0484	0.0421	0.0390	0.0270	0.0370	0.0476	0.0499
23	1	77	185	259	290	555	287	276	291	116	276	2613
Bias	0.0503	0.0812	0.0375	0.0149	-0.0001	0.0114	-0.0055	-0.0017	0.0076	0.0171	-0.0151	0.0082
Av Er	0.0503	0.0867	0.0640	0.0433	0.0311	0.0315	0.0250	0.0224	0.0211	0.0211	0.1477	0.0455
24	1	77	185	259	290	555	287	276	291	116	276	2613
Bias	-0.0119	0.0249	0.0021	-0.0042	-0.0040	0.0187	0.0074	0.0089	0.0091	0.0124	0.0150	0.0089
Av Er	0.0119	0.0448	0.0438	0.0382	0.0298	0.0330	0.0246	0.0232	0.0218	0.0194	0.1265	0.0403
25	1	77	185	259	290	555	287	276	291	116	276	2613
Bias	-0.0240	0.0292	0.0139	0.0051	-0.0015	0.0145	0.0001	0.0039	0.0108	0.0182	-0.0175	0.0058
Av Er	0.0240	0.0493	0.0445	0.0374	0.0295	0.0314	0.0244	0.0226	0.0221	0.0220	0.0684	0.0340

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