SAMPLE SIZE ESTIMATION
MADE EASY

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ABSTRACT

This paper describes briefly the principles of hypothesis testing and its associated error types, namely, type I and type II errors. The important role that sample size plays in the probability of committing either type of error is pointed out. A procedure that is easy to read and follow is described for determining the required sample size for most experimental purposes. The procedure is outlined first for cases in which the probability of obtaining a confidence interval less than or equal to a specified length or the probability of detecting a false hypothesis is not specified. A similar procedure is given for cases in which the probability of not exceeding a specified confidence interval length or the probability of detecting a false hypothesis is specified.

Several examples are worked out in detail to clarify the procedures of sample size estimation. Three tables provide the required sample size for a wide range of allowable errors and coefficients of variation and for the more commonly used significance levels. It is hoped that most researchers in forestry and related fields will find the procedures outlined here easier to apply and remember than those given in statistical textbooks.

RÉSUMÉ

Les auteurs décrivent brèvement les principes du test d'hypothèse et des erreurs types allant de pair: l'erreur type I et l'erreur type II. Ils insistent sur le rôle important que joue la grandeur de l'échantillon sur la probabilité de faire intervenir l'une ou l'autre erreur type. Ils décrivent une méthode, facile à lire et à suivre, qui permet de déterminer la grandeur de l'échantillon pour la plupart des expériences. La méthode tient compte d'abord des cas où la probabilité d'obtenir un intervale de confiance plus petit que ou égal à une longueur spécifiée ou où la probabilité de détecter une fausse hypothèse n'est pas donnée. Les auteurs suggèrent une méthode semblable dans les cas où la probabilité de ne pas excéder une longueur spécifiée d'intervale de confiance ou où la probabilité de détecter une fausse hypothèse est spécifiée.

Plusieurs exemples détaillés éclairent la méthode d'estimation de la grandeur de l'échantillon. Trois tableaux donnent la grandeur requise d'échantillon pour une large gamme d'erreurs acceptables et de coefficients de variation et pour les niveaux les plus communément utilisés de signification. La plupart des scientifiques forestiers et de champs d'activités connexes devraient trouver que les méthodes décrites ici seront plus faciles à appliquer et mémoriser que les méthodes fournies dans les manuels de statistique.

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INTRODUCTION

Biologists, foresters and other research professionals, when carrying out an experiment, are always confronted with the problem of determining sample size or the required number of replications. Sample size estimation is such a basic problem that it is discussed in almost every statistical textbook. Yet in practice many professionals find the procedures outlined in these textbooks ambiguous and difficult to apply. The purpose of this paper is to provide a procedure for determining sample size for experimental purposes that is easy to read and follow. Formulae for estimating sample size are explained and accompanied by detailed examples.

Two basic purposes of sampling are to estimate population parameters and to test assumptions about them without examining the entire population. One could make an intuitive assumption that, as sample size increases, the more the sample will resemble the population. However, this is true only up to a point, and that point depends on the inherent variability of the characteristic being estimated. Sampling too many units for a characteristic that varies only slightly throughout the population can be wasteful; on the other hand, sampling too few units for a highly variable characteristic would likely result in incorrect conclusions.

Sample size plays an important role in determining the probability of drawing wrong conclusions on the basis of the sample. Thus it is desirable first to examine briefly the principles of hypothesis testing and the role that sample size plays in it.

BACKGROUND INFORMATION

Let us assume that earth is the only planet capable of sustaining life, that artificial sweeteners will cause cancer in human beings, that Canadians prefer pine to spruce for Christmas trees, that fertilization always improves growth of forest stands, or that planting methods have no effect on tree survival and growth. These assumptions or *contentions* are usually based on some preliminary observations of what appear to be facts, but may or may not be true. In statistical jargon, the contentions are referred to as *hypotheses*. The test of a hypothesis is the comparison of a contention with newly and objectively collected facts, i.e., with a *sample*. If the sample is shown to support the contention, the hypothesis is accepted; if it does not, the hypothesis is rejected.

Since a sample contains only a fraction of the population, conclusions drawn from the testing of a hypothesis are not always correct. That is, one of two types of errors may be committed: either a hypothesis that is indeed true is rejected, or a hypothesis that is in fact false is accepted. The former is termed a type I error and the latter a type II error. This may be summarized as follows[1]:

Nature of hypothesis	Action to be taken	
	Acceptance	Rejection
true	correct conclusion	type I error
false	type II error	correct conclusion

Note that the two kinds of errors cannot be committed simultaneously. If a hypothesis is accepted, only a type II error can be committed. If the hypothesis is rejected, only a type I error can be committed. (Note that neither type of error will *necessarily* be committed.) Obviously, one would like to minimize these decision errors, or at the very least, be aware of the probability of committing them in hypothesis testing. Minimizing such decision errors is indeed a major problem because, for a given population and a given sample size, any attempt to decrease the risk of one error type is likely to increase the risk of incurring the other. The only way to decrease both is to increase the sample size, and of course this is not always possible.

In hypothesis testing, the probability that a type I error will occur is called the significance level and is denoted, hereafter, by the Greek letter alpha (α). In order to minimize bias, this probability should be decided upon before any samples are taken. For example, if a 5% significance level is used to test a certain hypothesis, then there is a 5% chance that, on the basis of the sample, the hypothesis will be rejected when it is really true; conversely, there is a 95% chance that, on the basis of the sample, a true hypothesis will be accepted. In practice, 5% and 1% significance levels are used most commonly, although in many cases 10% is considered adequate.

For any given sample size, the danger in making α as small as possible-- which on first glance would seem to be the thing to do-- is that as α is decreased, the chance of committing a type II error, or not detecting a false hypothesis, is increased. The probability of making a type II error is denoted, hereafter, by the Greek letter beta (β).

In general, the power of a test is the probability that it will reject a hypothesis which is indeed false. Therefore, the more

[1] After Li (1964).

powerful the test, the more likely it is that a false hypothesis will be refuted. The two types of errors are usually controlled by deciding first on an acceptable significance level and then increasing the sample size until the power of the test is acceptable.

In most statistical textbooks, the section dealing with sampling methods usually includes a statement to the effect that "it is assumed that the population follows the normal distribution and the sample is taken at random." Such a statement discourages most people for two reasons. In the first place, they do not know how the population is distributed; in the second place, for various reasons, it is simply impractical to take a strictly random sample. Actually, in most circumstances this is not a major problem. One of the most powerful theorems in statistics, the Central Limit Theorem, makes it possible to justify use of the sample mean as an estimate of the population mean, no matter how the population is distributed, as long as it has a finite variance and the sample size is large enough. Just how large is large enough will depend on how close to normally distributed the population is.[2] In rare cases the parent distribution may be such that the distribution of sample means may not converge to normal at all. In other cases, although the Central Limit Theorem will apply, sample size estimation based on the normality assumption may not be very efficient (Alvo 1977). However, for most natural populations the normality assumption will be satisfied for sample sizes of, say, 25 or larger. This is not to say that a sample size of 25 is large enough to satisfy other requirements discussed in the following section.

Samples should be random to include as much of the variation in the population as possible and to eliminate as much bias as possible. Therefore, if one has some knowledge of the population, one can ensure that a sufficient variety of units is included in the sample to offer a complete range of values for the measured characteristic (Cochran 1963). That is to say, as long as the sample has been made as unbiased or as representative of the population as possible, one need not worry whether it is strictly random or not.

SAMPLE SIZE ESTIMATION

The foregoing discussion should demonstrate the important role that sample size plays in the design of experiments. In this section, formulae and examples of sample size estimation will be given in three parts: first, the most common method in which the probability of

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[2] The closer a distribution is to the normal, the faster the rate of convergence of the distribution of sample means will be to normality.

obtaining a confidence interval $\leq$ a specified length is not specified, but lies in the neighborhood of .5 or 50% (see Harris et al. 1948 for explanation), i.e., this method would give about a 50-50 chance of detecting a false hypothesis; second, a slightly modified version of the first formula, and demonstration of a method in which the required sample size can be found from a set of tables; and third, formulae that incorporate the probability of obtaining a confidence interval $\leq$ a specified length or of detecting a false hypothesis, together with examples.

Part I

To calculate sample size (n), estimates of the following five values are needed: variance, length of confidence interval, significance level, degrees of freedom, and a t-value. Although some of these values may seem impossible to obtain at first, after some consideration reasonable estimates can usually be obtained.

1. Variance (s^2). As indicated earlier, an approximation of the variability of the characteristic being estimated is necessary for sample size determination. For normal populations this can be obtained by one of the following methods:

   a) from knowledge of the variance of this characteristic on the same population or on one that is similar

   b) by presampling, i.e., by taking a small random sample of the population of size n_1, and calculating the variance of the sample as:

$$s^2 = \frac{\Sigma y^2 - \frac{(\Sigma y)^2}{n_1}}{n_1 - 1}$$

where: y = the value of the measured characteristic on an observation

n_1 = the number of observations in the sample,

and using this sample variance as an estimate of the population variance. If the final sample size (n) calculated on the basis of this estimate is less than or equal to n_1, then n_1 is large enough and no additional sampling is warranted. However, if n is greater than n_1, only $(n - n_1)$ additional observations will be needed to complete the survey. This method is quite reliable, but not always possible.

c) by using the range. If the population is sufficiently large, e.g., if it contains more than 500 elements, and if the range of values can be estimated easily, then a crude estimate of variance may be calculated as:

$$s^2 = (\text{range}/4)^2$$

In the case of binomial populations (enumerative - belonging to one class or another), any of the above three methods may be used. However, for the second method, the sample variance would be calculated as:

$$s^2 = p\ (1-p)$$

where: p is an estimate of expected percentage, or proportion, of successes in the population.

A success may be a germinated seed, a dead insect, a stocked quadrat, presence of a disease, etc. The closer the estimate of p is to either 0% or 100%, the more accurate its estimate should be. In the absence of any reasonable estimate, p = .5 or 50% may be used to obtain the largest possible variance, and this will result in the largest possible sample size.

2. Confidence interval (±d). As mentioned earlier, one must be willing to accept some margin of error in the sample estimate. In scientific research, this margin is usually somewhat arbitrary, but careful consideration should be given to the way in which the results of the experiment are to be used and the possible consequences of a sizeable error. The interval in which the sample estimate is expected to be is the value of the population parameter plus or minus d. For example, if a confidence interval within 10% of the estimate appears reasonable, and the population parameter is estimated at 50, the confidence interval would be 50 ± 10%, or between 45 and 55, and d would be equal to 5.

3. Significance level (α). The significance level, α, is the risk one is willing to incur that the sample estimate will be outside the confidence interval. This is the probability of committing the type I error discussed earlier. It was also mentioned that α-values of 5% and 1% are most common.

4. Degrees of freedom (df). Degrees of freedom can be defined as the number of linearly independent comparisons that can be made from a set of observations. For most purposes, this is the sample size less the number of parameters being estimated. For the simple t-test, however, this is one less than the sample size. This value is used only to locate the t-value from the t-distribution table, and is critical only for small samples (fewer than 20 observations), or for confidence levels of less than 2%. Otherwise, it is sufficient to use $n = df = \infty$.

5. t - Value ($t_{(\alpha, df)}$). This value is found in a table of t - values (Appendix A). Each column of this table is for a given significance level. The rows correspond to different degrees of freedom. Note that if df is greater than 20 and α = 5%, then $t_{(.05, \infty)} \approx 2.0$. In practice, this value can be used without changing the sample size appreciably.

Now, with the estimates above, the following formula can be used to estimate the required sample size. Note that the standard deviation and confidence interval must be based on the same units, (i.e., metres, feet, proportions, percentages, etc.):

$$n = \frac{t^2_{(\alpha,\ df)}\ s^2}{d^2} \tag{1}$$

There are four common cases in which the above may not give the best estimate of the required sample size:

a) If the size of the population is relatively small (finite) and the calculated sample size n exceeds 10% of the population size N, then a finite population correction factor is applied to provide the final sample size estimate as:

$$n^* = \frac{n}{1 + \frac{n}{N}} \tag{2}$$

b) If the calculated sample size is small (less than 20), then, as stated earlier, it is necessary to estimate degrees of freedom to obtain a t-value. Because this is only an estimate, it may be best to recalculate the formula, using (n-1) degrees of freedom for the t-value. The iterative procedure (changing n and t-value) is continued until the n-values converge, i.e., until the formula approaches equality. This is necessary, because t-values change rapidly for small sample sizes. The number of iterations necessary will depend entirely on the closeness of the initial estimate to the required sample size.

c) If the population is highly heterogeneous, or if some portions of it are more costly to sample than others, it may be necessary to use stratified sampling. Then it is necessary to decide on the size of sample to be taken from each stratum. This is usually done in one of three ways: 1) samples of equal size may be taken from each stratum, 2) sample size may be made proportional to the size of each stratum (proportional allocation), or 3) larger sample size may be taken in a stratum if: i) the stratum is larger, ii) the stratum is more heterogeneous internally, and/or iii) sampling is cheaper in that stratum (optimum allocation). For further discussion on sampling allocation see Snedecor (1966, p. 504-512) and Cochran (1963, p. 90-98).

For most purposes, it is sufficient to solve the sample size formula (1) for each stratum.

d) In most surveys, estimation of more than one characteristic is considered. In such cases, the sample size may be calculated for the most important and most heterogeneous characteristics. If the two sample sizes do not differ greatly, the larger one can be used. If they do differ considerably, it may be possible to take the larger sample and measure the less heterogeneous characteristic(s) on a sub-sample only. If this is not possible, then the experiment may be broken up into separate surveys.

If, because of limited resources, the sample size may not exceed a maximum value, it is recommended that the length of the confidence interval be calculated from such a sample size as:

$$d = \left(\frac{t^2_{(\alpha,\ df)}\ s^2}{n} \right)^{.5} \qquad (3)$$

If d is too large for the experimental needs, then either a larger sample should be taken or the experiment should be discarded.

The following examples are given to clarify sample size estimation.

Example 1: A pulp and paper company has applied for a cutting license and an operational cruise must be conducted on the 100 hectare site to determine the merchantable volume/hectare of jack pine. The company needs to know how many plots (strips of 10 m wide and 40 m long, i.e., .04 ha) must be cruised to obtain a reasonable volume estimate for the entire site. Obviously, it is necessary to solve the sample size formula ($n = \frac{t^2 s^2}{d^2}$) for n.

The first problem is to estimate the variability of volume throughout the entire area. A forester recalls conducting a similar cruise on a similar jack pine stand several years earlier in which 200 plots (.04 ha) were taken. On that cruise, the plot volumes ranged from 6 m^3 to 24 m^3 and averaged close to 16 m^3. From this information, a rough estimate of plot variance would be: $s^2=(R/4)^2=(18/4)^2=20.25$.

The next step is to decide on a confidence interval. Because the volume estimate should be very close to what the company will actually harvest, it is decided that to be off by more than ± 10% of the mean volume in the final estimate, i.e., $d = .10 \times 16 = 1.6\ m^3$, is unacceptable. With a 5% significance level and a sample size greater than 20, t^2 will be approximately equal to 4. After these values have been substituted, the formula becomes:

$$n = \frac{4 \times 20.25}{(1.6)^2} = 31.64$$

From this, it is concluded that 32 plots (.04 ha) should be cruised for a reliable volume estimate. Because the total area to be sampled is far less than 10% of the stand, there is no need to apply the population correction factor.

Example 2. An entomologist needs to determine the spruce budworm density in a specified area. The standard sampling unit is a 45 cm branch tip taken from half way down the tree. The question is, how many branch tips should be taken to obtain a reliable density estimate for this area? The budworm density of a similar area was found to be 30 larvae/branch tip with a sample variance of 400.

Because of inherent variability and other factors affecting population density, it was decided that an error of ± 3 insects/branch tip would be quite acceptable at the 5% significance level. Since the sample size is expected to be greater than 20 branch tips, t^2 will be equal to 4. The required sample size will be:

$$n = \frac{t^2 s^2}{d^2} = \frac{4 \times 400}{9} \simeq 178.$$

However, owing to recent budget constraints, it is known that no more than 50 branch tips may be sampled in each area. Then, using this sample size for n, the formula is solved for d as:

$$50 = \frac{4 \times 400}{d^2} \text{ or } d^2 = 32 \text{ or } d = \pm\ 5.7 \simeq 6.$$

Because a confidence interval of 24-36 insects/branch tip is still acceptable under the circumstances, the project is continued.

Example 3: A forest pathologist has been assigned to study the impact of volume loss due to root rot in the mixedwood forests of northwestern Ontario. Several recent articles have indicated volume and growth loss of up to 30% in similar areas. Intensive sampling for root diseases is quite expensive since it involves root excavation, washing, cutting, transportation and laboratory analysis. Previous studies in this area have involved sampling of hundreds of trees. Because of resource limitations, the present researcher would like to take as small a sample as possible, yet be able to evaluate the loss due to root rot with a reasonable confidence. Because he is at a loss to where to begin, he consults a biometrician.

It is suggested that the researcher plan his work in several stages, concentrating first on the most vulnerable species and on trees of commercial size. On this basis he decides to work first on balsam fir trees $\geq$ 20 cm in diameter. Having defined his population of interest, he is asked if he has any idea what the intensity of root rot might be in merchantable-sized balsam fir trees. He states that he would not be surprised if up to 90% of the trees were affected by root rot. He claims he would be satisfied if he could estimate the percentage of infected trees within ± 5% and with 95% confidence.

To be on the safe side, we will assume that the intensity of infection is 50% or p = .5. As stated earlier this will result in maximum variance for a binomial population as:

$$s^2 = p\ (1-p) = .5 \times (1-.5) = .25.$$

Then the maximum sample size required may be calculated as:

$$n = \frac{2^2 \times .25}{(.05)^2} = \frac{1}{.0025} = 400$$

However, owing to the high cost of sampling it is suggested that the researcher first take a presample of perhaps 50 trees and then recalculate his required sample size. Suppose that 35 of the 50 trees were classified as severely infected by root rot, i.e., $p = 35 \div 50 = .7$ and $s^2 = .7 \times (1-.7) = .21$. In the meantime, the researcher discovers that his criterion for classifying the extent of root rot is somewhat subjective so he decides that the confidence interval of ± 5% is perhaps too rigid for such a criterion. Therefore, he raises his confidence interval to ± 10%. Now he can recalculate the required sample size as:

$$n = \frac{4 \times .21}{(.1)^2} = 84.$$

Since he has already taken a sample of 50 trees, he needs to take only 34 more. For other tree species in the stand that are less vulnerable to root rot, or for which the incidence is less frequent but more uniform, a smaller sample size is required, and this may be calculated in a similar manner.

Example 4: Two years ago, a cut-over area of 50 hectares was planted with 100,000 white spruce seedlings. Half of the site was scarified, and the seedlings were spot fertilized at planting with the equivalent of 100 kg/ha NPK. The other half of the site was used as a control area. The purpose of the experiment was to evaluate the effects of scarification and fertilization on the early growth of a white spruce plantation.

The researcher has no idea of the magnitude and the variability of growth differences between the treated and untreated plantations but does feel that growth should be higher in the fertilized area. Nevertheless, he would like to examine a sample large enough to detect a difference in growth as low as .2 g oven-dry weight of current foliage with 99% confidence.

Translating his requirement into the language of hypothesis testing, we say that he would like to test the hypothesis, "there is no difference in growth of trees between the fertilized and the control areas". He will accept this hypothesis if the sample difference is less than .2 g. Because his conclusion depends on this one sample, he wants to be 99% sure that he accepts the hypothesis if it is in fact true. Therefore, he chooses the significance level, α, or the probability of making a type I error of 1%. In this example, he does not specify the probability that the resulting confidence interval will be $\leq$ .2 g, or the probability of detecting a false hypothesis; therefore, he takes about a 50-50 chance of accepting the hypothesis when it is in fact false (see Harris et al. (1948) for explanation).

Under the conditions cited above he decides to take a presample of 25 seedlings, 10 from the control area and 15 from the treated area. The current foliage from the two sets of seedlings was clipped and oven-dried. The preliminary analysis showing the calculation of sample mean difference and an estimate of pooled variance is given in Table 1. The t-value for the 1% significance level and 23 degrees of freedom is 2.807. Therefore, the sample size formula becomes:

$$n = \frac{(2.807)^2 \times .351}{(.2)^2} \simeq 70.$$

Since 10 and 15 seedlings were presampled from the control and treated areas, 60 and 55 more seedlings need to be sampled from each respective area.

Part II

The formula for calculating sample size used in the examples above may be expressed in a slightly different form as follows:

$$n = \frac{t^2_{(\alpha,\ df)}(CV)^2}{(AE)^2} \qquad (4)$$

where: CV = coefficient of variation in percent = $(\frac{\sqrt{s^2}}{\text{mean}}) \times 100$

Table 1. Preliminary analysis of a sample of oven-dry weight of foliage from 10 and 15 white spruce seedlings in a control and a treated area, respectively.

Seedlings	Current yield, oven-dry weight (g)		Combination	Explanation
	Control (y_c)	Treated area (y_t)		
1	2.11	5.06		
2	3.65	5.12		
3	3.75	4.21		
4	3.82	5.22		
5	2.81	5.04		
6	4.15	4.01		
7	4.31	5.70		
8	3.96	5.35		
9	4.05	5.17		
10	3.72	5.00		
11		4.32		
12		5.12		
13		4.07		
14		4.82		
15		5.64		
Σy	36.33	73.85		
$\bar{y}$	3.63	4.92	-1.29	$\bar{y}_c - \bar{y}_t$
n	10	15		
SS	4.06	4.01	8.07	pooled SS
df.	9	14	23	pooled df.
s^2	0.45	.29	.35	$s_p^2 = 8.07 \div 23 = .351$
$^1/n$	.1	.067	.167	

AE = allowable error or confidence interval

as percent of the mean = $(\frac{d}{\text{mean}}) \times 100$

and other symbols as defined earlier.

The main difference between the formula above and the previous one (equation 1) is that an estimate of the mean as well as the variance is required to calculate the coefficient of variation or relative variability. An estimate of the mean may be obtained in a manner similar to that in which the variance estimate is obtained, i.e., either from knowledge of previous surveys on the same or similar populations or by presampling. The main advantage of equation 4, however, is that the required sample size may be tabulated for a wide range of relative variabilities (coefficients of variation) and allowable errors for each of the common significance levels. Three such tables were constructed and are given in Appendix B. These tables provide the required sample size for significance levels of 1%, 5% and 10%, respectively.

In addition to providing a required sample size, these tables might also be used to examine the effect of changes in sample size on the allowable error and vice versa for any given situation. If interpolation between table values is necessary, remember that the relationship between sample size and coefficient of variation and allowable error is not linear.

To demonstrate the use of equation 4 it is best to rework two of the foregoing examples. In example 1, the estimates for plot mean and variance were 16 m^3 and 20.25 respectively. The coefficient of variation is calculated as $\frac{\sqrt{20.25}}{16} \times 100$ or 28.13%. Using Table B2 for the 5% significance level, we locate the required sample size for 10% allowable error and 25% and 30% coefficient of variation, as 27 and 37, respectively. The required sample size for CV% of 28.13% will be roughly half way between these two values, or 32, as was calculated before. If an allowable error as large as ± 20% is acceptable, Table B2 indicates that the required sample will be between 9 and 11, or about 10. On the other hand, if the allowable error may not exceed ± 5%, then the required sample size will fall between 98 and 138, say 120. If the significance level is changed to 1%, Table B1 indicates that the required sample size for ± 10% allowable error falls between 46 and 64, or about 56, and for 15% allowable error, it will be between 22 and 30, or about 26. Thus, once an estimate of CV% is calculated, the required sample size for a range of allowable errors and/or significance levels may be found in the tables in Appendix B; or conversely, the effects that change in sample size will have on allowable error and/or significance level may be examined.

For example 2, the estimate for mean number of insects/branch was 30 with a variance of 400. The confidence interval was specified as ± 3 insects for a significance level of 5%. We may calculate CV% as $\frac{\sqrt{400}}{30} \times 100$ or 66.67% and AE% as $\pm \frac{3}{30} \times 100$ or ± 10%. Table B2 gives sample sizes of 138 and 188 for 60% and 70% coefficients of variation, for 5% significance level. By interpolation, we obtain the required sample size of 178 as before. If we reduce the sample size to 50, Table B2 indicates that the allowable error will be 20% as calculated earlier. Now, if we assume that even an allowable error of 20% is sufficient at a 10% significance level, then Table B3 indicates that the required sample size will be between 26 and 35, i.e., 30 branch tips/area. The two examples above should indicate clearly the usefulness of equation 4 and Tables B1-3.

Part III

The procedures and formulae for estimating sample size described so far do not specify the probability of obtaining a confidence interval $\leq$ a specified length or the probability of detecting a false hypothesis. For instance, in example 4, the calculated sample size of 70 trees would be large enough to detect a true difference of > .2 g if it had existed approximately 50% of the time, and would detect no difference, if there was indeed no true difference, 99% of the time. Many times, however, the experimenter needs much stronger assurance that he will find a difference in his sample if it does exist in the population, and he is willing to take a much larger sample to get it. Three general cases for doing so are given below:

1. Single population: The formula (Li 1964) for determining sample size required to estimate a single population parameter by a confidence interval no greater than a specified length is:

$$n = \frac{4\ s^2}{d^2}\ F_{\alpha(1,n-1)}\ F_{\beta(n-1,\nu)} \tag{5}$$

where: s^2 and d^2 are as defined earlier; F_α is the F-value found in Appendix C for α significance level, 1 degree of freedom for numerator, and (n-1) degrees of freedom for denominator; and F_β is the F-value found in Appendix C for β significance level, (n-1) degrees of freedom for numerator, and ν degrees of freedom for denominator, where ν = the degrees of freedom for the variance estimate. As stated earlier, α is the required level of significance, and β is the probability or the required degree of assurance that the length of confidence interval will not be exceeded. This formula is not as formidable as it looks. To illustrate this, we will use it in reworking example 1.

As in the case of equation 1 for small values of n, an iterative process is required to solve equation 5 because both F_α and F_β are dependent on a value for n. You will recall that the objective was to determine the number of plots required to estimate the merchantable volume of jack pine with ± 10% accuracy and 95% confidence (i.e., $\alpha = .05$). Assume further that the company now wants to guarantee with fairly good assurance, say in three out of four chances or 75%, that the volume estimate does not exceed the specified interval of ± 10% (i.e., $\beta = .25$). You will recall also that the variance estimate of 20.25 was obtained from a similar cruise based on 200 plots. Therefore, ν, or degrees of freedom for the variance estimate s^2, is (200-1) or 199. This information can be substituted in equation 5 as follows:

$$n = \frac{4 \times 20.25}{(.1 x 16)^2} F_{.05(1,n-1)} F_{.25(n-1,199)}$$

$$\text{or } n = 31.64 \times F_{.05(1,n-1)} \times F_{.25(n-1,199)}$$

Note that 31.64 was the required number of plots calculated using equation 1, when the probability of obtaining a confidence interval $\leq$ a desired length was not specified. Because the company requires stronger assurances that the desired interval will not be exceeded, it will have to take a larger sample; in fact, it will have to increase its sample size by the product of the two tabulated F-values.

To solve equation 5, we require a rough estimate for n, for example, 61. Now, the required F-values can be found in Appendix C. The F-value at the .05 significance level, and 1 and 60 degrees of freedom, is 4.0; the F-value at the .25 significance level, and 60 and 199 degrees of freedom, is approximately 1.14. When we solve for a new n-value, the equation becomes:

$$n = 31.64 \times 4.0 \times 1.14 \approx 144.$$

The estimate was too small (the estimate and calculated value for n should be approximately equal); consequently, the same procedure is followed using a new estimate closer to 144, say 141. If we use n = 141, the required F-values are $F_{.05(1,140)} \approx 3.9$ and $F_{.25(140,199)} \approx 1.1$.

When we solve again for n, the new calculated value becomes:

$$n = 31.64 \times 3.9 \times 1.1 \approx 136.$$

This time the estimate was slightly high; that is, the correct sample size will be between 136 and 141. However, as the F-tables indicate that F-values for degrees of freedom between 120 and ∞ change only slightly, a sample size of 138 can be used and another iteration is not necessary.

2. Two populations: The formula Li (1964) for determining the sample size n for the confidence interval of the difference between two population means is:

$$n = \frac{8\ s^2}{d^2} F_{\alpha(1,\ 2n-2)} F_{\beta(2n-2,\nu)} \tag{6}$$

All variables have the same interpretation as was given in the case of a single population. The application of this formula will be demonstrated by reworking example 4.

In addition to the requirements stated earlier, suppose that the researcher in example 4 also wants to be quite sure, with a probability of perhaps .95 or 19 out of 20 times, that his sample size will be large enough to detect a true difference of greater than .2 g if it does indeed exist. Because the presample was based on 25 seedlings from two different areas, ν will be (25-2) or 23. Therefore, for this example,

$s^2 = .351$ $\qquad d^2 = .04$ $\qquad \alpha = .01$

$\beta = .05$ $\qquad \nu = 23$

and $$n = \frac{8 \times .351}{.04} F_{.01(1,2n-2)} F_{.05(2n-2,23)},$$

which must be solved by trial and error. If n = 121 is used for the first trial, then (2n-2) is 240, and from the appropriate F-tables,

$$F_{.01(1,240)} \simeq 6.79 \text{ and } F_{.05(240,23)} \simeq 1.79$$

and $$n = 70.2 \times 6.79 \times 1.79 \simeq 853.$$

Note that the first portion of the above equation, i.e., excluding the F-values, results in n = 70, the same sample size as determined earlier. However, when the probability of obtaining a confidence interval $\leq .2$ or the probability of detecting a false hypothesis is specified at 95% ($\beta = .05$), a sample size approximately 12 times (the product of the two F-values, or $1.8 \times 6.8 = 12.24$) larger than before is required.

The calculation above indicates that the first estimate for n = 121 was too low. Since the F-values between 120 and ∞ degrees of freedom decrease only slightly, $n = \infty$ may be used as the second estimate, and this results in the minimum sample size required. The F-values for $F_{.01\ (1,\infty)} = 6.635$ and $F_{.05\ (\infty,23)} = 1.753$ are obtained from the F-tables and the equation becomes:

$$n = 70.2 \times 6.635 \times 1.753 \simeq 818.$$

Therefore, the required sample size lies between 818 to 853, but closer to the latter, say n = 850. That is, a random sample of 850 seedlings must be taken from each of the control and treated areas to meet the researcher's strict requirements.

If the researcher feels that he cannot afford to take samples as large as 850, he will have to lower his requirements by accepting a wider confidence interval, a lower significance level, a lower probability for detecting a difference within the specified limits, or finally a combination of all three. To illustrate this, we will assume for the example above that the confidence interval is increased to ± .4 g, the significance level α increased to .05, and the probability of detecting a difference greater than .4 g lowered to 90%, i.e., β = .10. Equation 6, when solved with the original estimate of 121, becomes:

$$n = \frac{8 \times .351}{.4^2} F_{.05\ (1,240)}\ F_{.10\ (240,23)}$$

or $\quad n = 17.55 \times 3.89 \times 1.58$

or $\quad n \simeq 108.$

This indicates that the initial value of 121 was large. If n = 111 is used as the second estimate, the solution of equation 6 for n gives:

$$n = 17.55 \times 3.91 \times 1.59 \simeq 110.$$

Therefore, the final estimate for n will fall between 110 and 111, and will be rounded off to 110. Since 10 and 15 seedlings were already sampled from the control and treated areas, respectively, 100 and 95 more seedlings will be needed from each respective area.

3. Several populations: For experiments involving more than two populations (treatments), the sample size formula is:

$$n = \frac{s^2\ Q^2_{\alpha(a,f)}\ F_{\beta(f,\nu)}}{d^2} \tag{7}$$

where a is the number of treatments requiring a sample size of n, f = the degrees of freedom for the error mean square in the planned experiment; $F_{\beta\ (f,\nu)}$ is obtained from the F-tables as before; and $Q_{\alpha\ (a,f)}$ is obtained from the Q-tables given in Appendix D. Other terms are as defined earlier. Note that if the planned experiment is a completely randomized design, f = a (n-1); if the planned experiment is a randomized block design, f = (a-1) (n-1).

To illustrate the application of this formula, example 4 will be expanded still further. Assume that the researcher is planning an experiment to determine the effects of four levels of NPK fertilizer (0, 100, 200, and 500 kg/ha) on the early growth of white spruce plantations in a completely randomized design. He would like to be able to detect growth differences, among the four treatments, of as low as .4 g (in oven-dry weight of current foliage) with .9 probability (β = .10) and at the 5% significance level (α = .05). He believes that the growth variation for the four treatments should not be much greater than that for the two treatments he had sampled previously, so he assumes that $s^2 = .4$ will be a reasonable variance estimate to use. The information on hand is then

$$s^2 = .4 \qquad d = .4 \qquad \nu = 23 \qquad a = 4$$

$$f = 4n-4 \qquad \alpha = .05 \qquad \beta = .10$$

If we use 31 as a first estimate for n, we obtain the values $F_{.10(120,23)} = 1.59$ and $Q_{.05(4,120)} = 3.69$ from the F and Q tables, respectively. Equation 7 now becomes:

$$n = \frac{.4 \times (3.69)^2 \times 1.59}{.16} \simeq 54.$$

This indicates that the first estimate was too low, and n = 51 is used for the second trial. Again, after we have obtained $F_{.10\ (200,23)} \simeq 1.575$ and $Q_{.05\ (4,200)} \simeq 3.67$ from the respective tables, and have substituted in the formula, equation 7 becomes:

$$n = \frac{.4 \times (3.67)^2 \times 1.575}{.16} \simeq 53.$$

The second estimate was quite close, i.e., the required sample size is between 51 and 53, but closer to 53. Therefore, 53 seedlings should be sampled from each of the four treated areas. With 53 replications, the researcher will have a 90% chance of detecting a growth difference as low as .4 g among the four treatments at the 5% significance level, if such a difference does indeed exist.

Exercise: Suppose that the researcher in example 4 is planning to expand his fertilization experiment further on a white spruce plantation. He would like to examine the effects of N, P, and K fertilizers on the early growth of seedlings measured in terms of oven-dry weight of current foliage as before. He is planning to use a completely randomized 3-factor factorial design with 4 levels of N (0, 250, 500 and 1000 kg/ha), 3 levels of P (0, 100, and 300 kg/ha), and 2 levels of K (100 and 200 kg/ha). He is interested in testing for the main effects and 2-way interactions, but not for the 3-way interaction at this time. Several recent papers on similar studies indicate that .6 will be a reasonable pooled variance estimate with perhaps 50 degrees of freedom, i.e., $s^2 = .6$ and $\nu = 50$. He would like to be able to detect differences in growth (due to the effects and interactions of fertilizers) of .5 g or greater (oven-dry weight) with 75% or three out of four times assurance, i.e., $\beta = 1-.75 = .25$. He is satisfied to use the 5% significance level as before. How many replications/fertilizer combinations does he need to satisfy his requirements for this experiment? If the researcher's resources are limited to a maximum of 20 replications (trees)/fertilizer combinations, what magnitude of growth differences will he be able to detect? The answers to these questions are given in Appendix E.

SUMMARY

For determining sample size or the number of replications required for a given experiment, the researcher must approximate the variability of the population(s) being studied. He may do this by using his knowledge of, or experience with, similar populations, or by pre-sampling. The researcher must also decide on a confidence interval, usually expressed as a percentage of the parameter being estimated. Once a significance level is decided upon, it is possible to obtain an estimate for the required sample size. This method provides modest assurance or about one out of two chances of detecting a false hypothesis. If greater assurance is required, it must also be specified. However, one must pay for this extra assurance with a much larger sample size. Formulae that incorporate the probability of obtaining a confidence interval less than or equal to a specified length or the probability of detecting a fale hypothesis are given for three general cases: a) a single population, b) two populations, and c) several populations.

If an experiment requires modest assurance for detecting a false hypothesis and if the variance and confidence intervals are expressed as percentages of the mean, then tables may be used to determine the sample size. These tables may also be used to examine the effects of change in confidence interval or significance level on the sample size and vice versa. If the calculated sample size is larger than can be accommodated, the formulae and tables may also be used to determine the confidence interval of a sample size that can be accommodated. However, if this confidence interval is not acceptable, the experiment should be postponed until more resources can be found.

Regardless of the formula used to determine sample size, it is essential that the population(s) or experimental materials be as homogeneous as possible (by stratification and laboratory techniques), and that as much bias as possible be eliminated. It is also wise to keep in mind that 5% chances really do occur--about 5 times in every 100.

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APPENDICES

APPENDIX A

Table A1. Percentage points of the t-distribution[a]

Degrees of freedom	Probability of a larger value, sign ignored .2(20%)	.1(10%)	.05(5%)	.02(2%)	.01(1%)
1	3.078	6.314	12.706	31.821	63.657
2	1.866	2.920	4.303	6.965	9.925
3	1.638	2.353	3.182	4.541	5.841
4	1.533	2.132	2.776	3.747	4.604
5	1.476	2.015	2.571	3.365	4.032
6	1.440	1.943	2.447	3.143	3.707
7	1.415	1.895	2.365	2.998	2.499
8	1.397	1.860	2.306	2.896	3.355
9	1.383	1.833	2.262	2.821	3.250
10	1.372	1.812	2.228	2.764	3.169
11	1.363	1.796	2.201	2.718	3.106
12	1.356	1.782	2.179	2.681	3.055
20	1.325	1.725	2.086	2.528	2.845
25	1.316	1.708	2.060	2.485	2.787
30	1.310	1.697	2.042	2.457	2.750
40	1.303	1.684	2.021	2.423	2.704
60	1.296	1.671	2.000	2.390	2.660
120	1.289	1.658	1.980	2.358	2.617
∞	1.282	1.645	1.960	2.326	2.576

[a]This table is reproduced, with the permission of Professor D. R. Cox, from Biometrika, vol. 32, p. 311.

APPENDIX B

Table B1. Required sample size for a range of allowable errors and coefficients of variation for a significance level of 1%.

	Confidence Interval or Allowable Error (%)												
CV%	1	2	3	4	5	10	15	20	30	40	50	75	100
1	10	5	4	4	4	3	3	3	2[a]	2	2	2	2
2	30	10	6	5	5	3	3	3	3	3	2	2	2
5	166	46	22	15	10	5	4	4	3	3	3	3	2
10	664	166	79	46	30	10	7	5	4	4	3	3	3
15	1493	373	166	96	64	19	10	7	5	4	4	3	3
20		664	295	166	110	30	16	10	7	5	5	4	3
25		1037	461	259	166	46	22	15	8	6	5	4	4
30		1493	664	373	239	64	30	19	10	7	6	5	4
35			903	508	325	87	40	24	13	10	7	5	4
40			1180	664	425	110	51	30	16	10	8	5	5
45			1493	840	537	134	64	37	19	12	9	6	5
50			1843	1037	664	166	79	46	22	15	10	7	5
60				1493	956	239	110	64	30	19	14	8	6
70					1301	325	145	87	40	24	17	10	7
80					1699	425	189	110	51	30	21	11	8
90	n > 2000					537	239	139	64	37	25	14	9
100						664	295	166	79	46	30	16	10
150						1493	664	373	166	96	64	30	19
200							1180	664	295	166	110	51	30
250							1843	1037	461	259	166	79	46
300								1493	664	373	239	110	64
400									1180	664	425	189	110
500									1843	1037	664	295	166

[a]Note that the minimum sample size required is 2 to provide at least one degree of freedom for statistical analysis.

(Continued)

Table B2. Required sample size for a range of allowable errors and coefficients of variation for a significance level of 5%.

	Confidence Interval or Allowable Error (%)												
CV%	1	2	3	4	5	10	15	20	30	40	50	75	100
1	6	4	3	3	3	2	2	2	2	2	2	2	2
2	17	6	4	4	3	3	2	2	2	2	2	2	2
5	98	27	13	8	6	4	3	3	2	2	2	2	2
10	384	98	45	7	17	6	4	3	3	3	3	2	2
15	864	216	98	56	37	11	6	5	4	3	3	3	2
20	1537	384	171	98	64	17	9	6	5	4	4	3	2
25		600	268	150	98	27	13	9	5	4	4	3	3
30		864	384	216	138	37	17	11	6	5	4	3	3
35		1176	523	294	188	50	24	15	8	6	4	4	3
40		1537	683	384	246	64	30	17	9	6	5	4	3
45		1945	864	486	311	81	37	22	11	8	6	4	3
50			1067	600	384	98	45	27	13	9	6	4	3
60			1537	864	553	138	64	37	17	11	8	5	4
70				1176	753	188	87	50	23	15	10	6	4
80				1537	983	246	112	64	30	17	13	7	5
90		n > 2000		1945	1245	311	138	81	37	22	15	8	6
100					1537	384	171	98	45	27	17	9	6
150						864	384	216	98	56	37	7	11
200						1537	683	384	171	98	64	30	17
250							1067	600	267	510	98	45	26
300							1537	864	384	216	138	64	37
400								1537	683	384	246	112	64
500									1067	600	384	171	98

(Continued)

Table B3. Required sample size for a range of allowable error and coefficients of variation for a confidence limit of 10%.

	Confidence Interval or Allowable Error (%)												
CV%	1	2	3	4	5	10	15	20	30	40	50	75	100
1	5	3	3	2	2	2	2	2	2	2	2	2	2
2	13	5	4	3	3	2	2	2	2	2	2	2	2
5	70	19	10	6	5	3	3	2	2	2	2	2	2
10	271	70	32	19	13	5	4	3	3	2	2	2	2
15	609	152	70	40	26	8	5	4	3	2	2	2	2
20	1082	271	122	70	45	13	7	5	4	3	3	2	2
25	1691	423	188	107	70	19	10	6	4	3	3	3	2
30		609	271	152	99	26	13	8	5	4	3	3	2
35		829	368	207	133	35	17	10	6	4	4	3	3
40		1082	481	271	173	45	21	13	7	5	4	3	3
45		1370	609	342	219	57	27	16	8	6	4	3	3
50		1691	752	423	271	70	32	19	9	6	5	4	3
60			1082	609	390	99	45	26	13	8	6	4	3
70			1473	829	530	133	61	35	17	10	8	5	3
80			1924	1082	693	173	79	45	21	13	9	5	4
90				1370	877	219	99	56	26	16	11	6	4
100		n > 2000		1691	1082	271	122	70	32	19	13	7	5
150						609	271	152	70	40	26	13	8
200						1082	481	271	122	70	45	21	13
250						1691	752	423	188	107	70	32	19
300							1082	609	271	152	99	45	26
400							1924	1082	481	271	173	79	45
500								1691	752	423	271	122	70

APPENDIX C

F - table[a]

Table C1. 25%, 10%, 5%, 2.5%, 1% and 0.5% points for the distribution of F

Denominator df	P	Numerator df 1	2	3	4	5	6	7	8	9	10	12	15	20	24	30	40	60	120	∞
1	.250	5.83	7.50	8.20	8.58	8.82	8.98	9.10	9.19	9.26	9.32	9.41	9.49	9.58	9.63	9.67	9.71	9.76	9.80	9.85
	.100	39.86	49.50	53.59	55.83	47.25	58.20	58.91	59.44	59.86	60.20	60.70	61.22	61.74	62.00	62.26	62.53	62.79	63.06	63.33
	.050	161.45	199.50	215.71	224.58	230.16	233.99	236.77	238.88	240.54	241.88	243.91	245.95	248.01	249.05	250.09	251.15	252.20	253.25	254.32
	.025	648	800	864	900	922	937	948	957	963	969	977	985	993	997	1,001	1,006	1,010	1,014	1,018
	.010	4,052	5,000	5,403	5,624	5,763	5,859	5,928	5,981	6,022	6,055	6,106	6,157	6,208	6,234	6,260	6,286	6,313	6,339	6,366
	.005	16,211	20,000	21,615	22,500	23,056	23,436	23,715	23,925	24,091	24,224	24,426	24,630	24,836	24,940	25,044	25,148	25,253	25,350	25,465
2	.250	2.57	3.00	3.15	3.23	3.28	3.31	3.34	3.35	3.37	3.38	3.39	3.41	3.43	3.43	3.44	3.45	3.46	3.47	3.48
	.100	8.53	9.00	9.16	9.24	9.29	9.33	9.35	9.37	9.38	9.39	9.41	9.42	9.44	9.45	9.46	9.47	9.47	9.48	9.49
	.050	18.51	19.00	19.16	19.25	19.30	19.33	19.35	19.37	19.39	19.40	19.41	19.43	19.45	19.45	19.46	19.47	19.48	19.49	19.50
	.025	38.51	39.00	39.16	39.25	39.30	39.33	39.36	39.37	39.39	39.40	39.42	39.43	39.45	39.46	39.46	39.47	39.48	39.49	39.50
	.010	98.50	99.00	99.17	99.25	99.30	99.33	99.36	99.37	99.39	99.40	99.42	99.43	99.45	99.46	99.47	99.47	99.48	99.49	99.50
	.005	198	199	199	199	199	199	199	199	199	199	199	199	199	199	199	199	199	199	200
3	.250	2.02	2.28	2.36	2.39	2.41	2.42	2.43	2.44	2.44	2.44	2.45	2.46	2.46	2.46	2.46	2.47	2.47	2.47	2.47
	.100	5.54	5.46	5.39	5.34	5.31	5.28	5.27	5.25	5.24	5.23	5.22	5.20	5.18	5.18	5.17	5.16	5.15	5.14	5.13
	.050	10.13	9.55	9.28	9.12	9.02	8.94	8.89	8.85	8.81	8.79	8.74	8.70	8.66	8.64	8.62	8.59	8.57	8.55	8.53
	.025	17.44	16.04	15.44	15.10	14.88	14.74	14.62	14.54	14.47	14.42	14.34	14.25	14.17	14.12	14.08	14.04	13.99	13.95	13.90
	.010	34.12	30.82	29.46	28.71	28.24	27.91	27.67	27.49	27.35	27.23	27.05	26.87	26.69	26.60	26.51	26.41	26.32	26.22	26.13
	.005	55.55	49.80	47.47	46.20	45.39	44.84	44.43	44.13	43.88	43.69	43.39	43.08	42.78	42.62	42.47	42.31	42.15	41.99	41.83
4	.250	1.81	2.00	2.05	2.06	2.07	2.08	2.08	2.08	2.08	2.08	2.08	2.08	2.08	2.08	2.08	2.08	2.08	2.08	2.08
	.100	4.54	4.32	4.19	4.11	4.05	4.01	3.98	3.95	3.94	3.92	3.90	3.87	3.84	3.83	3.82	3.80	3.79	3.78	3.76
	.050	7.71	6.94	6.59	6.39	6.26	6.16	6.09	6.04	6.00	5.96	5.91	5.86	5.80	5.77	5.75	5.72	5.69	5.66	5.63
	.025	12.22	10.65	9.98	9.60	9.36	9.20	9.07	8.98	8.90	8.84	8.75	8.66	8.56	8.51	8.46	8.41	8.36	8.31	8.26
	.010	21.20	18.00	16.69	15.98	15.52	15.21	14.98	14.80	14.66	14.55	14.37	14.20	14.02	13.93	13.84	13.75	13.65	13.56	13.46
	.005	31.33	26.28	24.26	23.16	22.46	21.98	21.62	21.35	21.14	20.97	20.70	20.44	20.17	20.03	19.89	19.75	19.61	19.47	19.32
5	.250	1.69	1.85	1.88	1.89	1.89	1.89	1.89	1.89	1.89	1.89	1.89	1.89	1.88	1.88	1.88	1.88	1.87	1.87	1.87
	.100	4.06	3.78	3.62	3.52	3.45	3.40	3.37	3.34	3.32	3.30	3.27	3.24	3.21	3.19	3.17	3.16	3.14	3.12	3.10
	.050	6.61	5.79	5.41	5.19	5.05	4.95	4.88	4.82	4.77	4.74	4.68	4.62	4.56	4.53	4.50	4.46	4.43	4.40	4.37
	.025	10.01	8.43	7.76	7.39	7.15	6.98	6.85	6.76	6.68	6.62	6.52	6.43	6.33	6.28	6.23	6.18	6.12	6.07	6.02
	.010	16.26	13.27	12.06	11.39	10.97	10.67	10.46	10.29	10.16	10.05	9.89	9.72	9.55	9.47	9.38	9.29	9.20	9.11	9.02
	.005	22.78	18.31	16.53	15.56	14.94	14.51	14.20	13.96	13.77	13.62	13.38	13.15	12.90	12.78	12.66	12.53	12.40	12.27	12.14
6	.250	1.62	1.76	1.78	1.79	1.79	1.78	1.78	1.78	1.77	1.77	1.77	1.76	1.76	1.75	1.75	1.75	1.74	1.74	1.74
	.100	3.78	3.46	3.29	3.18	3.11	3.05	3.01	2.98	2.96	2.94	2.90	2.87	2.84	2.82	2.80	2.78	2.76	2.74	2.72
	.050	5.99	5.14	4.76	4.53	4.39	4.28	4.21	4.15	4.10	4.06	4.00	3.94	3.87	3.84	3.81	3.77	3.74	3.70	3.67
	.025	8.81	7.26	6.60	6.23	5.99	5.82	5.70	5.60	5.52	5.46	5.37	5.27	5.17	5.12	5.07	5.01	4.96	4.90	4.85
	.010	13.75	10.93	9.78	9.15	8.75	8.47	8.26	5.10	7.98	7.87	7.72	7.56	7.40	7.31	7.23	7.14	7.06	6.97	6.88
	.005	18.64	14.54	12.92	12.03	11.46	11.07	10.79	10.57	10.39	10.25	10.03	9.81	9.59	9.47	9.36	9.24	9.12	9.00	8.88
7	.250	1.57	1.70	1.72	1.72	1.71	1.71	1.70	1.70	1.69	1.69	1.68	1.68	1.67	1.67	1.66	1.66	1.65	1.65	1.65
	.100	3.59	3.26	3.07	2.96	2.88	2.83	2.78	2.75	2.72	2.70	2.67	2.63	2.59	2.58	2.56	2.54	2.51	2.49	2.47
	.050	5.59	4.74	4.35	4.12	3.97	3.87	3.79	3.73	3.68	3.64	3.57	3.51	3.44	3.41	3.38	3.34	3.30	3.27	3.23
	.025	8.07	6.54	5.89	5.52	5.29	5.12	4.99	4.90	4.82	4.76	4.67	4.57	4.47	4.42	4.36	4.31	4.25	4.20	4.14
	.010	12.25	9.55	8.45	7.85	7.46	7.19	6.99	6.84	6.72	6.62	6.47	6.31	6.16	6.07	5.99	5.91	5.82	5.74	5.65
	.005	16.24	12.40	10.88	10.05	9.52	9.16	8.89	8.68	8.51	8.38	8.18	7.97	7.75	7.64	7.53	7.42	7.31	7.19	7.08

(cont'd)

APPENDIX C (cont'd)

F - table[a]

Table C1. 25%, 10%, 5%, 2.5%, 1% and 0.5% points for the distribution of F

Denominator df	P	1	2	3	4	5	6	7	8	9	10	12	15	20	24	30	40	60	120	∞
		Numerator df																		
8	.250	1.54	1.66	1.67	1.66	1.66	1.65	1.64	1.64	1.64	1.63	1.62	1.62	1.61	1.60	1.60	1.59	1.59	1.58	1.58
	.100	3.46	3.11	2.92	2.81	2.73	2.67	2.62	2.59	2.56	2.54	2.50	2.46	2.42	2.40	2.38	2.36	2.34	2.32	2.29
	.050	5.12	4.46	4.07	3.84	3.69	3.58	3.50	3.44	3.39	3.35	3.28	3.22	3.15	3.12	3.08	3.04	3.01	2.97	2.93
	.025	7.57	6.06	5.42	5.05	4.82	4.65	4.53	4.43	4.36	4.30	4.20	4.10	4.00	3.95	3.89	3.84	3.78	3.73	3.67
	.010	11.26	8.65	7.59	7.01	6.63	6.37	6.18	6.03	5.91	5.81	5.67	5.52	5.36	5.28	5.20	5.12	5.03	4.95	4.86
	.005	14.69	11.04	9.60	8.81	8.30	7.95	7.69	7.50	7.34	7.21	7.01	6.81	6.61	6.50	6.40	6.29	6.18	6.06	5.95
9	.250	1.51	1.62	1.63	1.63	1.62	1.61	1.60	1.60	1.59	1.59	1.58	1.57	1.56	1.56	1.55	1.54	1.54	1.53	1.53
	.100	3.36	3.01	2.81	2.69	2.61	2.55	2.51	2.47	2.44	2.42	2.38	2.34	2.30	2.28	2.25	2.23	2.21	2.18	2.16
	.050	5.12	4.26	3.86	3.63	3.48	3.37	3.29	3.23	3.18	3.14	3.07	3.01	2.94	2.90	2.86	2.83	2.69	2.75	2.71
	.025	7.21	5.71	5.08	4.72	4.48	4.32	4.20	4.10	4.03	3.96	3.87	3.77	3.67	3.61	3.56	3.51	3.45	3.39	3.33
	.010	10.56	8.02	6.99	6.42	6.60	5.80	5.61	5.47	5.35	5.26	5.11	4.96	4.81	4.73	4.65	4.57	4.48	4.40	4.31
	.005	13.61	10.11	8.72	7.96	7.47	7.13	6.88	6.69	6.54	6.42	6.23	6.03	5.83	5.73	5.62	5.52	5.41	5.30	5.19
10	.250	1.49	1.60	1.60	1.59	1.59	1.58	1.57	1.56	1.56	1.55	1.54	1.53	1.52	1.52	1.51	1.51	1.50	1.49	1.48
	.100	3.28	2.92	2.72	2.61	2.52	2.46	2.41	2.38	2.35	2.32	2.28	2.24	2.20	2.18	2.16	2.13	2.11	2.08	2.06
	.050	4.96	4.10	3.71	3.48	3.33	3.22	3.14	3.07	3.02	2.98	2.91	2.85	2.77	2.74	2.70	2.66	2.62	2.58	2.54
	.025	6.94	5.46	4.83	4.47	4.24	4.07	3.95	3.85	3.78	3.72	3.62	3.52	3.42	3.37	3.31	3.26	3.20	3.14	3.08
	.010	10.04	7.56	6.55	5.99	5.64	5.39	5.20	5.06	4.94	4.85	4.71	4.56	4.41	4.33	4.25	4.17	4.08	4.00	3.91
	.005	12.83	9.43	8.08	7.34	6.87	6.54	6.30	6.12	5.97	5.85	5.66	5.47	5.27	5.17	5.07	4.97	4.86	4.75	4.64
11	.250	1.47	1.58	1.58	1.57	1.56	1.55	1.54	1.53	1.53	1.52	1.51	1.50	1.49	1.49	1.48	1.47	1.47	1.46	1.45
	.100	3.23	2.86	2.66	2.54	2.45	2.39	2.34	2.30	2.27	2.25	2.21	2.17	2.12	2.10	2.08	2.05	2.03	2.00	1.97
	.050	4.84	3.98	3.59	3.36	3.20	3.09	3.01	2.95	2.90	2.85	2.79	2.72	2.65	2.61	2.57	2.53	2.49	2.45	2.40
	.025	6.72	5.26	4.63	4.28	4.04	3.88	3.76	3.66	3.59	3.53	3.43	3.33	3.23	3.17	3.12	3.06	3.00	2.94	2.88
	.010	9.65	7.21	6.22	5.67	5.32	5.07	4.89	4.74	4.63	4.54	4.40	4.25	4.10	4.02	3.94	3.86	3.78	3.69	3.60
	.005	12.23	8.91	7.60	6.88	6.42	6.10	5.86	5.68	5.53	5.42	5.24	5.05	4.86	4.76	4.65	4.55	4.44	4.34	4.23
12	.250	1.46	1.56	1.56	1.55	1.54	1.53	1.52	1.51	1.51	1.50	1.49	1.48	1.47	1.46	1.45	1.45	1.44	1.43	1.42
	.100	3.18	2.81	2.61	2.48	2.39	2.33	2.28	2.21	2.21	2.19	2.15	2.10	2.06	2.04	2.01	1.99	1.96	1.93	1.90
	.050	4.75	3.89	3.49	3.26	3.11	3.00	2.91	2.85	2.80	2.75	2.69	2.62	2.54	2.51	2.47	2.43	2.38	2.34	2.30
	.025	6.55	5.10	4.47	4.12	3.89	3.73	3.61	3.51	3.44	3.37	3.28	3.18	3.07	3.02	2.96	2.91	2.85	2.79	2.72
	.010	9.33	6.93	5.95	5.41	5.06	4.82	4.64	4.50	4.39	4.30	4.16	4.01	3.86	3.78	3.70	3.62	3.54	3.45	3.36
	.005	11.75	8.51	7.23	6.52	6.07	5.76	5.52	5.35	5.20	5.09	4.91	4.72	4.53	4.43	4.33	4.23	4.12	4.01	3.90
13	.250	1.45	1.55	1.55	1.53	1.52	1.51	1.50	1.49	1.49	1.48	1.47	1.46	1.45	1.44	1.43	1.42	1.42	1.41	1.40
	.100	3.14	2.76	2.56	2.43	2.35	2.28	2.23	2.20	2.16	2.14	2.10	2.05	2.01	1.98	1.96	1.93	1.90	1.88	1.85
	.050	4.67	3.81	3.41	3.18	3.03	2.92	2.83	2.77	2.71	2.67	2.60	2.53	2.46	2.42	2.38	2.34	2.30	2.25	2.21
	.025	6.41	4.97	4.35	4.00	3.77	3.60	3.48	3.39	3.31	3.25	3.15	3.05	2.95	2.89	2.84	2.78	2.72	2.66	2.60
	.010	9.07	6.70	5.74	5.21	4.86	4.62	4.44	4.30	4.19	4.10	3.96	3.82	3.66	3.59	3.51	3.43	3.34	3.25	3.17
	.005	11.37	8.19	6.93	6.23	5.79	5.48	5.25	5.08	4.94	4.82	4.64	4.46	4.27	4.17	4.07	3.97	3.87	3.76	3.65
14	.250	1.44	1.53	1.53	1.52	1.51	1.50	1.49	1.48	1.47	1.46	1.45	1.44	1.43	1.42	1.41	1.41	1.40	1.39	1.38
	.100	3.10	2.73	2.52	2.39	2.31	2.24	2.19	2.15	2.12	2.10	2.05	2.01	1.96	1.94	1.91	1.89	1.86	1.83	1.80
	.050	4.60	3.74	3.34	3.11	2.96	2.85	2.76	2.70	2.65	2.60	2.53	2.46	2.39	2.35	2.31	2.27	2.22	2.18	2.13
	.025	6.30	4.86	4.24	3.89	3.66	3.50	3.38	3.29	3.21	3.15	3.05	2.95	2.84	2.79	2.73	2.67	2.61	2.55	2.49
	.010	8.86	6.51	5.56	5.04	4.70	4.46	4.28	4.14	4.03	3.94	3.80	3.66	3.51	3.43	3.35	3.27	3.18	3.09	3.00
	.005	11.06	7.92	6.68	6.00	5.56	5.26	5.03	4.86	4.72	4.60	4.43	4.25	4.06	3.96	3.86	3.76	3.66	3.55	3.44

(cont'd)

APPENDIX C (cont'd)

F - table[a]

Table C1. 25%, 10%, 5%, 2.5%, 1% and 0.5% points for the distribution of F

Denom-inator df	P	Numerator df 1	2	3	4	5	6	7	8	9	10	12	15	20	24	30	40	60	120	∞
15	.250	1.43	1.52	1.52	1.51	1.49	1.48	1.47	1.46	1.46	1.45	1.44	1.43	1.41	1.41	1.40	1.39	1.38	1.37	1.36
	.100	3.07	2.70	2.49	2.36	2.27	2.21	2.16	2.12	2.09	2.06	2.02	1.97	1.92	1.90	1.87	1.85	1.82	1.79	1.76
	.050	4.54	3.68	3.29	3.06	2.90	2.79	2.71	2.64	2.59	2.54	2.48	2.40	2.33	2.29	2.25	2.20	2.16	2.11	2.07
	.025	6.20	4.76	4.15	3.80	3.58	3.41	3.29	3.20	3.12	3.06	2.96	2.86	2.76	2.70	2.64	2.58	2.52	2.46	2.40
	.010	8.68	6.36	5.42	4.89	4.56	4.32	4.14	4.00	3.89	3.80	3.67	3.52	3.37	3.29	3.21	3.13	3.05	2.96	2.87
	.005	10.80	7.70	6.48	5.80	5.37	5.07	4.85	4.67	4.54	4.42	4.25	4.07	3.88	3.79	3.69	3.58	3.48	3.37	3.26
16	.250	1.42	1.51	1.51	1.50	1.48	1.47	1.46	1.45	1.44	1.44	1.43	1.41	1.40	1.39	1.38	1.37	1.36	1.35	1.34
	.100	3.05	2.67	2.46	2.33	2.24	2.18	2.13	2.09	2.06	2.03	1.99	1.94	1.89	1.87	1.84	1.81	1.78	1.75	1.72
	.050	4.49	3.63	3.24	3.01	2.85	2.74	2.66	2.59	2.54	2.49	2.42	2.35	2.28	2.24	2.19	2.15	2.11	2.06	2.01
	.025	6.12	4.69	4.08	3.73	3.50	3.34	3.22	3.12	3.05	2.99	2.89	2.79	2.68	2.63	2.57	2.51	2.45	2.38	2.32
	.010	8.53	6.23	5.29	4.77	4.44	4.20	4.03	3.89	3.78	3.69	3.55	3.41	3.26	3.18	3.10	3.02	2.93	2.84	2.75
	.005	10.58	7.51	6.30	5.64	5.21	4.91	4.69	4.52	4.38	4.27	4.10	3.92	3.73	3.64	3.54	3.44	3.33	3.22	3.11
17	.250	1.42	1.51	1.50	1.49	1.47	1.46	1.45	1.44	1.43	1.43	1.41	1.40	1.39	1.38	1.37	1.36	1.35	1.34	1.33
	.100	3.03	2.64	2.44	2.31	2.22	2.15	2.10	2.06	2.03	2.00	1.96	1.91	1.86	1.84	1.81	1.78	1.75	1.72	1.69
	.050	4.45	3.59	3.20	2.96	2.81	2.70	2.61	2.55	2.49	2.45	2.38	2.31	2.23	2.19	2.15	2.10	2.06	2.01	1.96
	.025	6.04	4.62	4.01	3.66	3.44	3.28	3.16	3.06	2.98	2.92	2.82	2.72	2.62	2.56	2.50	2.44	2.38	2.32	2.25
	.010	8.40	6.11	5.19	4.67	4.34	4.10	3.93	3.79	3.68	3.59	3.46	3.31	3.16	3.08	3.00	2.92	2.83	2.75	2.65
	.005	10.38	7.35	6.16	5.50	5.07	4.78	4.56	4.39	4.25	4.14	3.97	3.79	3.61	3.51	3.41	3.31	3.21	4.10	2.98
18	.250	1.41	1.50	1.49	1.48	1.46	1.45	1.44	1.43	1.42	1.42	1.40	1.39	1.38	1.37	1.36	1.35	1.34	1.33	1.32
	.100	3.01	2.62	2.42	2.29	2.20	2.13	2.08	2.04	2.00	1.98	1.93	1.89	1.84	1.81	1.78	1.75	1.72	1.69	1.66
	.050	4.41	3.55	3.16	2.93	2.77	2.66	2.58	2.51	2.46	2.41	2.34	2.27	2.19	2.15	2.11	2.06	2.02	1.97	1.92
	.025	5.98	4.56	3.95	3.61	3.38	3.22	3.10	3.01	2.93	2.87	2.77	2.67	2.56	2.50	2.44	2.38	2.32	2.26	2.19
	.010	8.29	6.01	5.09	4.58	4.25	4.01	3.84	3.71	3.60	3.51	3.37	3.23	3.08	3.00	2.92	2.84	2.75	2.66	2.57
	.005	10.22	7.21	6.03	5.37	4.96	4.66	4.44	4.28	4.14	4.03	3.86	3.68	3.50	3.40	3.30	3.20	3.10	2.99	2.87
19	.250	1.41	1.49	1.49	1.47	1.46	1.44	1.43	1.42	1.41	1.41	1.40	1.38	1.37	1.36	1.35	1.34	1.33	1.32	1.30
	.100	2.99	2.61	2.40	2.27	2.18	2.11	2.06	2.02	1.98	1.96	1.91	1.86	1.81	1.79	1.76	1.73	1.70	1.67	1.63
	.050	4.38	3.52	3.13	2.90	2.74	2.63	2.54	2.48	2.42	2.38	2.31	2.23	2.16	2.11	2.07	2.03	1.98	1.93	1.88
	.025	5.92	4.51	3.90	3.56	3.33	3.17	3.05	2.96	2.88	2.82	2.72	2.62	2.51	2.45	2.39	2.33	2.27	2.20	2.13
	.010	8.19	5.93	5.01	4.50	4.17	3.94	3.77	3.63	3.52	3.43	3.30	3.15	3.00	2.92	2.84	2.76	2.67	2.58	2.49
	.005	10.07	7.09	5.92	5.27	4.85	4.56	4.34	4.18	4.04	3.93	3.76	3.59	3.40	3.31	3.21	3.11	3.00	2.89	2.78
20	.250	1.40	1.49	1.48	1.47	1.45	1.44	1.43	1.42	1.41	1.40	1.39	1.37	1.36	1.35	1.34	1.33	1.32	1.31	1.29
	.100	2.97	2.59	2.38	2.25	2.16	2.09	2.04	2.00	1.96	1.94	1.89	1.84	1.79	1.77	1.74	1.71	1.68	1.64	1.61
	.050	4.35	3.49	3.10	3.87	2.71	2.60	2.51	2.45	2.39	2.35	2.28	2.20	2.12	2.08	2.04	1.99	1.95	1.90	1.84
	.025	5.87	4.46	3.86	3.51	3.29	3.13	3.01	2.91	2.84	2.77	2.68	2.57	2.46	2.41	2.35	2.20	2.22	2.16	2.09
	.010	8.10	5.85	4.94	4.43	4.10	3.87	3.70	3.56	3.46	3.37	3.23	3.09	2.94	2.86	2.78	2.69	2.61	2.52	2.42
	.005	9.94	6.99	5.82	5.17	4.76	4.47	4.26	4.09	3.96	3.85	3.68	3.50	3.32	3.22	3.12	3.02	2.92	2.81	2.69
21	.250	1.40	1.48	1.48	1.46	1.44	1.43	1.42	1.41	1.40	1.39	1.38	1.37	1.35	1.34	1.33	1.32	1.31	1.30	1.28
	.100	2.96	2.57	2.36	2.23	2.14	2.08	2.02	1.98	1.95	1.92	1.88	1.83	1.78	1.75	1.72	1.69	1.66	1.62	1.59
	.050	4.32	3.47	3.07	2.84	2.68	2.57	2.49	2.42	2.37	2.32	2.25	2.18	2.10	2.05	2.01	1.96	1.92	1.87	1.81
	.025	5.83	4.42	3.82	3.48	3.25	3.09	2.97	2.87	2.80	2.73	2.64	2.53	2.42	2.37	2.31	2.25	2.18	2.11	2.04
	.010	8.02	5.78	4.87	4.37	4.04	3.81	3.64	3.51	3.40	3.31	3.17	3.03	2.88	2.80	2.72	2.64	2.55	2.46	2.36
	.005	9.83	6.89	5.73	5.09	4.68	4.39	4.18	4.01	3.88	3.77	3.60	3.43	3.24	3.15	3.05	2.95	2.84	2.73	2.61

(cont'd)

APPENDIX C (cont'd)

F - table[a]

Table C1. 25%, 10%, 5%, 2.5%, 1% and 0.5% points for the distribution of F

Denominator df	P	Numerator df 1	2	3	4	5	6	7	8	9	10	12	15	20	24	30	40	60	120	∞
22	.250	1.40	1.48	1.47	1.45	1.44	1.42	1.41	1.40	1.39	1.39	1.37	1.36	1.34	1.33	1.32	1.31	1.30	1.29	1.28
	.100	2.95	2.56	2.35	2.22	2.13	2.06	2.01	1.97	1.93	1.90	1.86	1.81	1.76	1.73	1.70	1.67	1.64	1.60	1.57
	.050	4.30	3.44	3.05	2.82	2.66	2.55	2.46	2.40	2.34	2.30	2.23	2.15	2.07	2.03	1.98	1.94	1.89	1.84	1.78
	.025	5.79	4.38	3.78	3.44	3.22	3.05	2.93	2.84	2.76	2.70	2.60	2.50	2.39	2.33	2.27	2.21	2.14	2.08	2.00
	.010	7.95	5.72	4.82	4.31	3.99	3.76	3.59	3.45	3.35	3.26	3.12	2.98	2.83	2.75	2.67	2.58	2.50	2.40	2.31
	.005	9.73	6.81	5.65	5.02	4.61	4.32	4.11	3.94	3.81	3.70	3.54	3.36	3.18	3.08	2.98	2.88	2.77	2.66	2.55
23	.250	1.39	1.47	1.47	1.45	1.43	1.42	1.41	1.40	1.39	1.38	1.37	1.35	1.34	1.33	1.32	1.31	1.30	1.28	1.27
	.100	2.94	2.55	2.34	2.21	2.11	2.05	1.99	1.95	1.92	1.89	1.84	1.80	1.74	1.72	1.69	1.66	1.62	1.59	1.55
	.050	4.28	3.42	3.03	2.80	2.64	2.53	2.44	2.37	2.32	2.27	2.20	2.13	2.05	2.01	1.96	1.91	1.86	1.81	1.76
	.025	5.75	4.35	3.75	3.41	3.18	3.02	2.90	2.81	2.73	2.67	2.57	2.47	2.36	2.30	2.24	2.18	2.11	2.04	1.97
	.010	7.88	5.66	4.76	4.26	3.94	3.71	3.54	3.41	3.30	3.21	3.07	2.93	2.78	2.70	2.62	2.54	2.45	2.35	2.26
	.005	9.63	6.73	5.58	4.95	4.54	4.26	4.05	3.88	3.75	3.64	3.47	3.30	3.12	3.02	2.92	2.82	2.71	2.60	2.48
24	.250	1.39	1.47	1.46	1.44	1.43	1.41	1.40	1.39	1.38	1.38	1.36	1.35	1.33	1.32	1.31	1.30	1.29	1.28	1.26
	.100	2.93	2.54	2.33	2.19	2.10	2.04	1.98	1.94	1.91	1.88	1.83	1.78	1.73	1.70	1.67	1.64	1.61	1.57	1.53
	.050	4.26	3.40	3.01	2.78	2.62	2.51	2.42	2.36	2.30	2.25	2.18	2.11	2.03	1.98	1.94	1.89	1.84	1.79	1.73
	.025	5.72	4.32	3.72	3.38	3.15	2.99	2.87	2.78	2.70	2.64	2.54	2.44	2.33	2.27	2.21	2.15	2.08	2.01	1.94
	.010	7.82	5.61	4.72	4.22	3.90	3.67	3.50	3.36	3.26	3.17	3.03	2.89	2.74	2.66	2.58	2.49	2.40	2.31	2.21
	.005	9.55	6.66	5.52	4.89	4.49	4.20	3.99	3.83	3.69	3.59	3.42	3.25	3.06	2.97	2.87	2.77	2.66	2.55	2.43
25	.250	1.39	1.47	1.46	1.44	1.42	1.41	1.40	1.39	1.38	1.37	1.36	1.34	1.33	1.32	1.31	1.29	1.28	1.27	1.25
	.100	2.92	2.53	2.32	2.18	2.09	2.02	1.97	1.93	1.89	1.87	1.82	1.77	1.72	1.69	1.66	1.63	1.59	1.56	1.52
	.050	4.24	3.39	2.99	2.76	2.60	2.49	2.40	2.34	2.28	2.24	2.16	2.09	2.01	1.96	1.92	1.87	1.82	1.77	1.71
	.025	5.69	4.29	3.69	3.35	3.13	2.97	2.85	2.75	2.68	2.61	2.51	2.41	2.30	2.24	2.18	2.12	2.05	1.98	1.91
	.010	7.77	5.57	4.68	4.18	3.86	3.63	3.46	3.32	3.22	3.13	2.99	2.85	2.70	2.62	2.54	2.45	2.36	2.27	2.17
	.005	9.48	6.60	5.46	4.84	4.43	4.15	3.94	3.78	3.64	3.54	3.37	3.20	3.01	2.92	2.82	2.72	2.61	2.50	2.38
26	.250	1.38	1.46	1.45	1.44	1.42	1.41	1.39	1.38	1.37	1.37	1.35	1.34	1.32	1.31	1.30	1.29	1.28	1.26	1.25
	.100	2.91	2.52	2.31	2.17	2.08	2.01	1.96	1.92	1.88	1.86	1.81	1.76	1.71	1.68	1.65	1.61	1.58	1.54	1.50
	.050	4.23	3.37	2.98	2.74	2.59	2.47	2.39	2.32	2.27	2.22	2.15	2.07	1.99	1.95	1.90	1.85	1.80	1.75	1.69
	.025	5.66	4.27	3.67	3.33	3.10	2.94	2.82	2.73	2.65	2.59	2.49	2.39	2.28	2.22	2.16	2.09	2.03	1.95	1.88
	.010	7.72	5.53	4.64	4.14	3.82	3.59	3.42	3.29	3.18	3.09	2.96	2.82	2.66	2.58	2.50	2.42	2.33	2.23	2.13
	.005	9.41	6.54	5.41	4.79	4.38	4.10	3.89	3.73	3.60	3.49	3.33	3.15	2.97	2.87	2.77	2.67	2.56	2.45	2.33
27	.250	1.38	1.46	1.45	1.43	1.42	1.40	1.39	1.38	1.37	1.36	1.35	1.33	1.32	1.31	1.30	1.28	1.27	1.26	1.24
	.100	2.90	2.51	2.30	2.17	2.07	2.00	1.95	1.91	1.87	1.85	1.80	1.75	1.70	1.67	1.64	1.60	1.57	1.53	1.49
	.050	4.21	3.35	2.96	2.73	2.57	2.46	2.37	2.31	2.25	2.20	2.13	2.06	1.97	1.93	1.88	1.84	1.79	1.73	1.67
	.025	5.63	4.24	3.65	3.31	3.08	2.92	2.80	2.71	2.63	2.57	2.47	2.36	2.25	2.19	2.13	2.07	2.00	1.93	1.85
	.010	7.68	5.49	4.60	4.11	3.78	3.56	3.39	3.26	3.15	3.06	2.93	2.78	2.63	2.55	2.47	2.38	2.29	2.20	2.10
	.005	9.34	6.49	5.36	4.74	4.34	4.06	3.85	3.69	3.56	3.45	3.28	3.11	2.93	2.83	2.73	2.63	2.52	2.41	2.29
28	.250	1.38	1.46	1.45	1.43	1.41	1.40	1.39	1.38	1.37	1.36	1.34	1.33	1.31	1.30	1.29	1.28	1.27	1.25	1.24
	.100	2.89	2.50	2.29	2.16	2.06	2.00	1.94	1.90	1.87	1.84	1.79	1.74	1.69	1.66	1.63	1.59	1.56	1.52	1.48
	.050	4.20	3.34	2.95	2.71	2.56	2.45	2.36	2.29	2.24	2.19	2.12	2.04	1.96	1.91	1.87	1.82	1.77	1.71	1.65
	.025	5.61	4.22	3.63	3.29	3.06	2.90	2.78	2.69	2.61	2.55	2.45	2.34	2.23	2.17	2.11	2.05	1.98	1.91	1.83
	.010	7.64	5.45	4.57	4.07	3.75	3.53	3.36	3.23	3.12	3.03	2.90	2.75	2.60	2.52	2.44	2.35	2.26	2.17	2.06
	.005	9.28	6.44	5.32	4.70	4.30	4.02	3.81	3.65	3.52	3.41	3.25	3.07	2.89	2.79	2.69	2.59	2.48	2.37	2.25

(cont'd)

APPENDIX C (conc'd)

F - table[a]

Table C1. 25%, 10%, 5%, 2.5%, 1% and 0.5% points for the distribution of F

Denom-inator df	P	Numerator df 1	2	3	4	5	6	7	8	9	10	12	15	20	24	30	40	60	120	∞
29	.250	1.38	1.45	1.45	1.43	1.41	1.40	1.38	1.37	1.36	1.35	1.34	1.32	1.31	1.30	1.29	1.27	1.26	1.25	1.23
	.100	2.89	2.50	2.28	2.15	2.06	1.99	1.93	1.89	1.86	1.83	1.78	1.73	1.68	1.65	1.62	1.58	1.55	1.51	1.47
	.050	4.18	3.33	2.93	2.70	2.55	2.43	2.35	2.28	2.22	2.18	2.10	2.03	1.94	1.90	1.85	1.81	1.75	1.70	1.64
	.025	5.59	4.20	3.61	3.27	3.04	2.88	2.76	2.67	2.59	2.53	2.43	2.32	2.21	2.15	2.09	2.03	1.96	1.89	1.81
	.010	7.60	5.42	4.54	4.04	3.73	3.50	3.33	3.20	3.09	3.00	2.87	2.73	2.57	2.49	2.41	2.33	2.23	2.14	2.03
	.005	9.23	6.40	5.28	4.66	4.26	3.98	3.77	3.61	3.48	3.38	3.21	3.04	2.86	2.76	2.66	2.56	2.45	2.33	2.21
30	.250	1.38	1.45	1.44	1.42	1.41	1.39	1.38	1.37	1.36	1.35	1.34	1.32	1.30	1.29	1.28	1.27	1.26	1.24	1.23
	.100	2.88	2.49	2.28	2.14	2.05	1.98	1.93	1.88	1.85	1.82	1.77	1.72	1.67	1.64	1.61	1.57	1.54	1.50	1.46
	.050	4.17	3.32	2.92	2.69	2.53	2.42	2.33	2.27	2.21	2.16	2.09	2.01	1.93	1.89	1.84	1.79	1.74	1.68	1.62
	.025	5.57	4.18	3.59	3.25	3.03	2.87	2.75	2.65	2.57	2.51	2.41	2.31	2.20	2.14	2.07	2.01	1.94	1.87	1.79
	.010	7.56	5.39	4.51	4.02	3.70	3.47	3.30	3.17	3.07	2.98	2.84	2.70	2.55	2.47	2.39	2.30	2.21	2.11	2.01
	.005	9.18	6.35	5.24	4.62	4.23	3.95	3.74	3.58	3.45	3.34	3.18	3.01	2.82	2.73	2.63	2.52	2.42	2.30	2.18
40	.250	1.36	1.44	1.42	1.40	1.39	1.37	1.36	1.35	1.34	1.33	1.31	1.30	1.28	1.26	1.25	1.24	1.22	1.21	1.19
	.100	2.84	2.44	2.23	2.09	2.00	1.93	1.87	1.83	1.79	1.76	1.71	1.66	1.61	1.57	1.54	1.51	1.47	1.42	1.38
	.050	4.08	3.23	2.84	2.61	2.45	2.34	2.25	2.18	2.12	2.08	2.00	1.92	1.84	1.79	1.74	1.69	1.64	1.58	1.51
	.025	5.42	4.05	3.46	3.13	2.90	2.74	2.62	2.53	2.45	2.39	2.29	2.18	2.07	2.01	1.94	1.88	1.80	1.72	1.64
	.010	7.31	5.18	4.31	3.83	3.51	3.29	3.12	2.99	2.89	2.80	2.66	2.52	2.37	2.29	2.20	2.11	2.02	1.92	1.80
	.005	8.83	6.07	4.98	4.37	3.99	3.71	3.51	3.35	3.22	3.12	2.95	2.78	2.60	2.50	2.40	2.30	2.18	2.06	1.93
60	.250	1.35	1.42	1.41	1.38	1.37	1.35	1.33	1.32	1.31	1.30	1.29	1.27	1.25	1.24	1.22	1.21	1.19	1.17	1.15
	.100	2.79	2.39	2.18	2.04	1.95	1.87	1.82	1.77	1.74	1.71	1.66	1.60	1.54	1.51	1.48	1.44	1.40	1.35	1.29
	.050	4.00	3.15	2.76	2.53	2.37	2.25	2.17	2.10	2.04	1.99	1.92	1.84	1.75	1.70	1.65	1.59	1.53	1.47	1.39
	.025	5.29	3.93	3.34	3.01	2.79	2.63	2.51	2.41	2.33	2.27	2.17	2.06	1.94	1.88	1.82	1.74	1.67	1.58	1.48
	.010	7.08	4.98	4.13	3.65	3.34	3.12	2.95	2.82	2.72	2.63	2.50	2.35	2.20	2.12	2.03	1.94	1.84	1.73	1.60
	.005	8.49	5.80	4.73	4.14	3.76	3.49	3.29	3.13	3.01	2.90	2.74	2.57	2.39	2.29	2.19	2.08	1.96	1.83	1.69
120	.250	1.34	1.40	1.39	1.37	1.35	1.33	1.31	1.30	1.29	1.28	1.26	1.24	1.22	1.21	1.19	1.18	1.16	1.13	1.10
	.100	2.75	2.35	2.13	1.99	1.90	1.82	1.77	1.72	1.68	1.65	1.60	1.54	1.48	1.45	1.41	1.37	1.32	1.26	1.19
	.050	3.92	3.07	2.68	2.45	2.29	2.18	2.09	2.02	1.96	1.91	1.83	1.75	1.66	1.61	1.55	1.50	1.43	1.35	1.25
	.025	5.15	3.80	3.23	2.89	2.67	2.52	2.39	2.30	2.22	2.16	2.05	1.94	1.82	1.76	1.69	1.61	1.53	1.43	1.31
	.010	6.85	4.79	3.95	3.48	3.17	2.96	2.79	2.66	2.56	2.47	2.34	2.19	2.03	1.95	1.86	1.76	1.66	1.53	1.38
	.005	8.18	5.54	4.50	3.92	3.55	3.28	3.09	2.93	2.81	2.71	2.54	2.37	2.19	2.09	1.98	1.87	1.75	1.61	1.43
∞	.250	1.32	1.39	1.37	1.35	1.33	1.31	1.29	1.28	1.27	1.25	1.24	1.22	1.19	1.18	1.16	1.14	1.12	1.08	1.00
	.100	2.71	2.30	2.08	1.94	1.85	1.77	1.72	1.67	1.63	1.60	1.55	1.49	1.42	1.38	1.34	1.30	1.24	1.17	1.00
	.050	3.84	3.00	2.60	2.37	2.21	2.10	2.01	1.94	1.88	1.83	1.75	1.67	1.57	1.52	1.46	1.39	1.32	1.22	1.00
	.025	5.02	3.69	3.12	2.79	2.57	2.41	2.29	2.19	2.11	2.05	1.94	1.83	1.71	1.64	1.57	1.48	1.39	1.27	1.00
	.010	6.63	4.60	3.78	3.32	3.02	2.80	2.64	2.51	2.41	2.32	2.18	2.04	1.88	1.79	1.70	1.59	1.47	1.32	1.00
	.005	7.88	5.30	4.28	3.72	3.35	3.09	2.90	2.74	2.62	2.52	2.36	2.19	2.00	1.90	1.79	1.67	1.53	1.36	1.00

[a] Reproduced from "Tables of percentage points of the invented beta (F) distribution" by M. Merrington and C. Thompson, Biometrika, 33:77 (1943) by permission of the authors and the editor.

APPENDIX D

Q - table[a]

Table D1. Upper 5% and 1% Percentage Points of the Studentized Range

Degrees of freedom f	α	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20		Error df
		a = number of treatments																				
5	.05	3.64	4.60	5.22	5.67	6.03	6.33	6.58	6.80	6.99	7.17	7.32	7.47	7.60	7.72	7.83	7.93	8.03	8.12	8.21	.05	5
	.01	5.70	6.97	7.80	8.42	8.91	9.32	9.67	9.97	10.24	10.48	10.70	10.89	11.08	11.24	11.40	11.55	11.68	11.81	11.93	.01	
6	.05	3.46	4.34	4.90	5.31	5.63	5.89	6.12	6.32	6.49	6.65	6.79	6.92	7.03	7.14	7.24	7.34	7.43	7.51	7.59	.05	6
	.01	5.24	6.33	7.03	7.56	7.97	8.32	8.61	8.87	9.10	9.30	9.49	9.65	9.81	9.95	10.08	10.21	10.32	10.43	10.54	.01	
7	.05	3.34	4.16	4.68	5.06	5.36	5.61	5.82	6.00	6.16	6.30	6.43	6.55	6.66	6.76	6.85	6.94	7.02	7.09	7.17	.05	7
	.01	4.95	5.92	6.54	7.01	7.37	7.68	7.94	8.17	8.37	8.55	8.71	8.86	9.00	9.12	9.24	9.35	9.46	9.55	9.65	.01	
8	.05	3.26	4.04	4.53	4.89	5.17	5.40	5.60	5.77	5.92	6.05	6.18	6.29	6.39	6.48	6.57	6.65	6.73	6.80	6.87	.05	8
	.01	4.74	5.63	6.20	6.63	6.96	7.24	7.47	7.68	7.87	8.03	8.18	8.31	8.44	8.55	8.66	8.76	8.85	8.94	9.03	.01	
9	.05	3.20	3.95	4.42	4.76	5.02	5.24	5.43	5.60	5.74	5.87	5.98	6.09	6.19	6.28	6.36	6.44	6.51	6.58	6.64	.05	9
	.01	4.60	5.43	5.96	6.35	6.66	6.91	7.13	7.32	7.49	7.65	7.78	7.91	8.03	8.13	8.23	8.32	8.41	8.49	8.57	.01	
10	.05	3.15	3.88	4.33	4.65	4.91	5.12	5.30	5.46	5.60	5.72	5.83	5.93	6.03	6.11	6.20	6.27	6.34	6.40	6.47	.05	10
	.01	4.48	5.27	5.77	6.14	6.43	6.67	6.87	7.05	7.21	7.36	7.48	7.60	7.71	7.81	7.91	7.99	8.07	8.15	8.22	.01	
11	.05	3.11	3.82	4.26	4.57	4.82	5.03	5.20	5.35	5.49	5.61	5.71	5.81	5.90	5.99	6.06	6.14	6.20	6.26	6.33	.05	11
	.01	4.39	5.14	5.62	5.97	6.25	6.48	6.67	6.84	6.99	7.13	7.25	7.36	7.46	7.56	7.65	7.73	7.81	7.88	7.95	.01	
12	.05	3.08	3.77	4.20	4.51	4.75	4.95	5.12	5.27	5.40	5.51	5.62	5.71	5.80	5.88	5.95	6.03	6.09	6.15	6.21	.05	12
	.01	4.32	5.04	5.50	5.84	6.10	6.32	6.51	6.67	6.81	6.94	7.06	7.17	7.26	7.36	7.44	7.52	7.59	7.66	7.73	.01	
13	.05	3.06	3.73	4.15	4.45	4.69	4.88	5.05	5.19	5.32	5.43	5.53	5.63	5.71	5.79	5.86	5.93	6.00	6.05	6.11	.05	13
	.01	4.26	4.96	5.40	5.73	5.98	6.19	6.37	6.53	6.67	6.79	6.90	7.01	7.10	7.19	7.27	7.34	7.42	7.48	7.55	.01	
14	.05	3.03	3.70	4.11	4.41	4.64	4.83	4.99	5.13	5.25	5.36	5.46	5.55	5.64	5.72	5.79	5.85	5.92	5.97	6.03	.05	14
	.01	4.21	4.89	5.32	5.63	5.88	6.08	6.26	6.41	6.54	6.66	6.77	6.87	6.96	7.05	7.12	7.20	7.27	7.33	7.39	.01	
15	.05	3.01	3.67	4.08	4.37	4.60	4.78	4.94	5.08	5.20	5.31	5.40	5.49	5.58	5.65	5.72	5.79	5.85	5.90	5.96	.05	15
	.01	4.17	4.83	5.25	5.56	5.80	5.99	6.16	6.31	6.44	6.55	6.66	6.76	6.84	6.93	7.00	7.07	7.14	7.20	7.26	.01	
16	.05	3.00	3.65	4.05	4.33	4.56	4.74	4.90	5.03	5.15	5.26	5.35	5.44	5.52	5.59	5.66	5.72	5.79	5.84	5.90	.05	16
	.01	4.13	4.78	5.19	5.49	5.72	5.92	6.08	6.22	6.35	6.46	6.56	6.66	6.74	6.82	6.90	6.97	7.03	7.09	7.15	.01	
17	.05	2.98	3.63	4.02	4.30	4.52	4.71	4.86	4.99	5.11	5.21	5.31	5.39	5.47	5.55	5.61	5.68	5.74	5.79	5.84	.05	17
	.01	4.10	4.74	5.14	5.43	5.66	5.85	6.01	6.15	6.27	6.38	6.48	6.57	6.66	6.73	6.80	6.87	6.94	7.00	7.05	.01	
18	.05	2.97	3.61	4.00	4.28	4.49	4.67	4.82	4.96	5.07	5.17	5.27	5.35	5.43	5.50	5.57	5.63	5.69	5.74	5.79	.05	18
	.01	4.07	4.70	5.09	5.38	5.60	5.79	5.94	6.08	6.20	6.31	6.41	6.50	6.58	6.65	6.72	6.79	6.85	6.91	6.96	.01	

(cont'd)

APPENDIX D (concl'd)

Q - table[a]

Table D1. Upper 5% and 1% Percentage Points of the Studentized Range

		a = number of treatments																			
Degrees of freedom f	α	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	α	Error df
19	.05	2.96	3.59	3.98	4.25	4.47	4.65	4.79	4.92	5.04	5.14	5.23	5.32	5.39	5.46	5.53	5.59	5.65	5.70	5.75	.05	19
	.01	4.05	4.67	5.05	5.33	5.55	5.73	5.89	6.02	6.14	6.25	6.34	6.43	6.51	6.58	6.64	6.72	6.79	6.84	6.89	.01	
20	.05	2.95	3.58	3.96	4.23	4.45	4.62	4.77	4.90	5.01	5.11	5.20	5.28	5.36	5.43	5.49	5.55	5.61	5.66	5.71	.05	20
	.01	4.02	4.64	5.02	5.29	5.51	5.69	5.84	5.97	6.09	6.19	6.29	6.37	6.45	6.52	6.59	6.64	6.71	6.76	6.82	.01	
24	.05	2.92	3.53	3.90	4.17	4.37	4.54	4.68	4.81	4.92	5.01	5.10	5.18	5.25	5.32	5.38	5.44	5.50	5.54	5.59	.05	24
	.01	3.96	4.54	4.91	5.17	5.37	5.54	5.69	5.81	5.92	6.02	6.11	6.19	6.26	6.33	6.39	6.45	6.51	6.56	6.61	.01	
30	.05	2.89	3.49	3.84	4.10	4.30	4.46	4.60	4.72	4.83	4.92	5.00	5.08	5.15	5.21	5.27	5.33	5.38	5.43	5.48	.05	30
	.01	3.89	4.45	4.80	5.05	5.24	5.40	5.54	5.65	5.76	5.85	5.93	6.01	6.08	6.14	6.20	6.26	6.31	6.36	6.41	.01	
40	.05	2.86	3.44	3.79	4.04	4.23	4.39	4.52	4.63	4.74	4.82	4.91	4.98	5.05	5.11	5.16	5.22	5.27	5.31	5.36	.05	40
	.01	3.82	4.37	4.70	4.93	5.11	5.27	5.39	5.50	5.60	5.69	5.77	5.84	5.90	5.96	6.02	6.07	6.12	6.17	6.21	.01	
60	.05	2.83	3.40	3.74	3.98	4.16	4.31	4.44	4.55	4.65	4.73	4.81	4.88	4.94	5.00	5.06	5.11	5.16	5.20	5.24	.05	60
	.01	3.76	4.28	4.60	4.82	4.99	5.13	5.25	5.36	5.45	5.53	5.60	5.67	5.73	5.79	5.84	5.89	5.93	5.98	6.02	.01	
120	.05	2.80	3.36	3.69	3.92	4.10	4.24	4.36	4.48	4.56	4.64	4.72	4.78	4.84	4.90	4.95	5.00	5.05	5.09	5.13	.05	120
	.01	3.70	4.20	4.50	4.71	4.87	5.01	5.12	5.21	5.30	5.38	5.44	5.51	5.56	5.61	5.66	5.71	5.75	5.79	5.83	.01	
∞	.05	2.77	3.31	3.63	3.86	4.03	4.17	4.29	4.39	4.47	4.55	4.62	4.68	4.74	4.80	4.84	4.89	4.93	4.97	5.01	.05	∞
	.01	3.64	4.12	4.40	4.60	4.76	4.88	4.99	5.08	5.16	5.23	5.29	5.35	5.40	5.45	5.49	5.54	5.57	5.61	5.65	.01	

[a] Reproduced from "Tables for Statisticians", Biometrika Vol. 1, 1954 with permission of the editors, and original author, J.M. May.

APPENDIX E

Solution to the exercise: As in the last example, the sample size formula for several populations should be solved for n by trial and error. The information given may be summarized as:

$$s^2 = .6 \qquad d = .5 \qquad \nu = 50$$

$$\alpha = .05 \qquad \beta = .25 \qquad a = ? \qquad f = ?$$

As stated earlier, a is the number of treatments to be compared. In this case it will be equal to the product of the number of levels in each of the three factors, i.e., 4 x 3 x 2 = 24. The value of f will be the degree of the error term of the analysis of variance. Using an initial estimate of, say, n = 5 replicates/treatment, the value of f may be found by subtraction in the analysis of variance table as follows:

i) degrees of freedom for total SS = 4 x 3 x 2 x 5 - 1 = 119

ii) degrees of freedom for the main effect and the two-way interactions = 3 + 2 + 1 + 6 + 3 + 2 = 17.

iii) f = error degrees of freedom = 119 - 17 = 102

The sample size formula then becomes:

$$n = \frac{.6 \times Q^2_{.05\ (24,102)}\ F_{.25\ (102,50)}}{.25}$$

The equation above can be solved by substituting $Q_{.05}$ (24,102) ≃ 5.15 and $F_{.25}$ (102,50) ≃ 1.18 obtained from appropriate tables:

$$n = \frac{.6 \times (5.15)^2 \times 1.18}{.25} \simeq 75$$

Obviously, the first estimate for n was too small. However, since both values of Q and F degrees of freedom larger than 120 decrease only slightly, no further iteration is necessary and the final estimate will be 75 replications (trees)/treatment combination.

To answer the second question, n is set to 20 and the equation above is solved for d as:

$$d = \left[\frac{6 \times (5.15)^2 \times 1.18}{20}\right]^{.5} = \pm .96 \text{ g.}$$

That is, with 20 replications/treatment combination, only a difference of .96 g or larger between treatment means may be detected with .75 probability and at the 5% significance level.